# Worksheet

11/26/2018

Consider a system of linear differential equations

$$\begin{cases} x_1' = 2x_1 - 4x_2 \\ x_2' = x_1 - 3x_2 \end{cases}$$

under the initial conditions $x_1(0) = 1$ and $x_2(0) = -2$.

(a) Put

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Write the system in matrix form $X' = AX$.

$$X' = \underbrace{\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}}_{A} X$$

(b) Diagonalize matrix $A$.

characteristic poly. $= \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2)$

$\lambda_1 = 1, \quad \lambda_2 = -2$

* Find $v_1$:

$$A - \lambda_1 I = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

* Find $v_2$:

$$A - \lambda_2 I = \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

(c) Use the formula

$$X(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1} X(0)$$

to find $X(t)$.

$$P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} , \quad X(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$X(t) = \underbrace{\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix}}_{\begin{bmatrix} 4e^{t} & e^{-2t} \\ e^{t} & e^{-2t} \end{bmatrix}} \frac{1}{3} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\begin{bmatrix} 3 \\ -9 \end{bmatrix}} = \begin{bmatrix} 4e^{t} & e^{-2t} \\ e^{t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{t} - 3e^{-2t} \\ e^{t} - 3e^{-2t} \end{bmatrix}$$

(d) Find explicit form of $x_1(t)$ and $x_2(t)$.

$$x_1(t) = 4e^{t} - 3e^{-2t}$$

$$x_2(t) = e^{t} - 3e^{-2t}$$