Some review problems for Final Exam

1. Diagonalize the matrix
$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

2. Let A be the matrix given in Problem 1. Find an explicit formula for A^n.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the mirror reflection with respect to the plane $2x + y - z = 0$.
   (a) What is the matrix representing f in standard basis?
   (b) What is the image of the point $(3, -4, 1)$ under this transformation?

4. Determine all values of a, b, c such that matrix
$$A = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$$
is diagonalizable.

5. Determine all values of a such that matrix
$$B = \begin{bmatrix} 1 & a & 1 \\ 0 & 1 & a \\ 0 & 0 & 3 \end{bmatrix}$$
is diagonalizable.

6. Determine all solutions of the following system. If there are infinitely many solutions, write them in parametric vector form.
$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 & = & 1 \\ 2x_1 + x_3 + x_4 & = & 0 \\ x_1 + 2x_2 + x_3 + 2x_4 & = & 1 \\ x_1 + x_2 & = & -1 \end{cases}$$

7. Consider the vectors
$$\begin{aligned} v_1 &= (1,\ 3,\ 1) \\ v_2 &= (1,\ 1,\ -1) \\ v_3 &= (-3,\ -4,\ 2) \\ v_4 &= (1,\ 0,\ -2) \end{aligned}$$
   (a) Is vector $v = (2, 3, 1)$ a linear combination of v_1, v_2, v_3, v_4 ?
   (b) Find a basis for the vector space spanned by v_1, v_2, v_3, v_4.
   (c) Do these vectors span $\mathbb{R}^3$? Explain your answer.

8. Let
$$A = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & -1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Find all vectors x satisfying the equation $Ax = b$. If there are infinitely many solutions, write them in parametric vector form.

9. Let x_0, x_1, x_2, x_3,... be a sequence defined recursively as follows:

$$\begin{cases} x_0 = 1 \\ x_1 = 1 \\ x_{n+1} = x_n + 2x_{n-1} \end{cases}$$

Find an explicit formula for x_n in terms of n.

10. Let

$$\begin{aligned} v_1 &= (1, 0, 1) \\ v_2 &= (1, 0, 0) \\ v_3 &= (2, 1, 1) \end{aligned}$$

Consider a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} f(v_1) &= v_1 - v_2 + 2v_3 \\ f(v_2) &= 2v_1 - v_2 \\ f(v_3) &= 2v_1 - v_2 \end{aligned}$$

(a) Check if $S = \{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^3$.

(b) Find $f(1, 2, 3)$.

(c) Find the matrix representing f in standard basis.

(d) Is f injective? Explain why.

(e) Is f surjective? Explain why.

11. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 0 & 5 \\ 3 & 2 & 0 & 5 & 2 \end{bmatrix}$$

(a) Determine the null space of A, a basis and the dimension.

(b) Determine the column space of A, a basis and the dimension.

(c) Determine the row space of A, a basis and the dimension.

Note: *As a convention, the trivial space* $\{0\}$ *is 0-dimensional, having basis* $\emptyset$ *(the empty set).*

12. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$,

$$f(x, y) = (x + y, x + 1, 2y)$$

Is f a linear map? Explain your answer.

13. Determine all values of c such that the following system is consistent, i.e. having at least one solution.

$$\begin{cases} x_1 + x_2 + cx_3 + x_4 &= c \\ -x_2 + x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 - x_4 &= -c \end{cases}$$