Problem 2 of HW2:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We get 4 equations of 4 unknows a, b, c, d:

$$\begin{cases} a + 2b = 0 \\ 2a + 4b = 0 \\ c + 2d = 0 \\ 2c + 4d = 0 \end{cases} \qquad \text{Augmented matrix: } \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Nonpivot columns are 2^{nd} and 4^{th} columns. Thus, b and d are free variables. Put $b = \alpha_1$ and $d = \alpha_2$. Solve the system from bottom to top, we get: $c = -2d = -2\alpha_2$, $a = -2b = -2\alpha_1$.

Therefore,

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2\alpha_1 & \alpha_1 \\ -2\alpha_2 & \alpha_2 \end{bmatrix} = \begin{bmatrix} -2\alpha_1 & \alpha_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2\alpha_2 & \alpha_2 \end{bmatrix}$$

splitting α_1 from α_2

$$= \alpha_1 \underbrace{\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}}_{B_1} + \alpha_2 \underbrace{\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}}_{B_2}$$