Homework Set 3

Due 10/12/2018

1. Let
$$A = \begin{bmatrix} 1 & 0 \\ 3 & -5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix}.$$
Determine
   (a) $2A - 3B$,
   (b) $(A+B)^2$,
   (c) $A^2 + 2AB + B^2$.

2. Let A and B be two square matrices of the same size. Show that the identities
$$\begin{aligned} (A+B)^2 &= A^2 + 2AB + B^2 \\ A^2 - B^2 &= (A-B)(A+B) \end{aligned}$$
hold true *if and only if* $AB = BA$ (i.e. A and B commute with each other.)
Hint: write $(A+B)^2 = (A+B)(A+B)$.

3. Determine if each following matrix is invertible. If so, find the inverse matrix.

   (a)
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

4. Recall that an invertible matrix A is always row equivalent to the identity matrix (denoted by $A \sim I_n$). As a consequence, A can be expressed as a product of elementary matrices. (An elementary matrix is a matrix obtained from the identity matrix I_n by performing on I_n *one* elementary row operation.) Factor the matrix in Part (a) and (b) of Problem 3 into elementary matrices.

5. Consider a linear transformation on the plane $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 3x + 5y \end{bmatrix}.$$
Let S be the square with vertices at
$$Q_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
Determine the inverse image of S under the linear transformation. In other words, what is the region R such that $f(R) = S$?

6. Find the determinant of the following matrices. Show the steps needed to get the answer.

(a)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & -5 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 1 & 1 & 1 \\ 3 & 6 & 5 & 4 \\ 3 & 3 & 2 & 2 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 3 & 2 & 2 \\ 2 & 1 & 3 & 3 \end{bmatrix}$$