Solutions to Prob 2b, 3b and 4a of HW6

2b) $f(x,y,z) = x+2y+z$ is linear.

Check:

- Additivity:

$$\begin{aligned} f(x_1+x_2, y_1+y_2, z_1+z_2) &= (x_1+x_2) + 2(y_1+y_2) + (z_1+z_2) \\ &= (x_1+2y_1+z_1) + (x_2+2y_2+z_2) \\ &= f(x_1,y_1,z_1) + f(x_2,y_2,z_2). \end{aligned}$$

- Multiplicative:

$$f(cx, cy, cz) = cx + 2cy + cz = c(x+2y+z) = cf(x,y,z)$$

3b) The matrix representing g is

$$A = \begin{bmatrix} | & | & | \\ g(e_1) & g(e_2) & g(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 8 \end{bmatrix}$$

Find inverse of A:

$$[A \mid I_3] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & -4 & 5 & -2 \\ 0 & 0 & 1 & \underbrace{5}_{} & -3 & 1 \end{array}\right] \quad \underbrace{\qquad\qquad}_{A^{-1}}$$

A^{-1} is the matrix representing g^{-1} (the inverse map of g).

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & -2 & 1 \\ -4 & 5 & -2 \\ 5 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y+z \\ -4x+5y-2z \\ 5x-3y+z \end{bmatrix}$$

$$g(x,y,z) = (-2y+z, -4x+5y-2z, 5x-3y+z).$$

4a) The matrix representing f is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ -2 & 0 & -4 & 1 \end{bmatrix}$$

The RREF of A is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The third column is nonpivot. Thus, $x_3 = t$ (free variable).

The last row : $x_4 = 0$

The second row: $x_2 = t$

the first row : $x_1 = -2t$

Thus,

$$\ker(f) = \left\{ t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$\ker(f)$ is a 1-dimensional subspace of $\mathbb{R}^4$, with basis $\{(-2, 1, 1, 0)\}$.