Homework Set 7

Due 11/09/2018

Note: In this homework set, you are allowed to use calculator or Matlab to compute RREF. However, for other tasks such as finding inverse or computing determinant, you need to write in detail every step leading up to the point where you compute RREF.

1. Consider the linear map

$$f : \mathbb{R}^3 \to \mathbb{R}^4, \quad f(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ 2x_1 + 4x_3,\ 3x_1 + 2x_2 + 4x_3,\ 5x_2 - 5x_3).$$

   (a) Determine $\ker(f)$ by finding a basis. What is its dimension?

   (b) Determine $\text{range}(f)$ by finding a basis. What is its dimension?

2. Determine all values of c such that the following linear map is bijective (i.e. both injective and surjective):

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (x + 2y - z,\ x + (c+2)y - z,\ x + 2y + cz).$$

3. Check if each following map is a linear map. If it is, explain why (by verifying the 2 criteria). If it is not, show how one of these criteria is violated.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x^2,\ y^2)$.

   (c) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (2x + 2y,\ x - z)$.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that:

$$\begin{aligned} f(2,\ 3,\ 1) &= (1,\ 0) \\ f(1,\ 0,\ 1) &= (2,\ -1) \\ f(-1,\ -2,\ 0) &= (-1,\ 1) \end{aligned}$$

   (a) Find the matrix representing f.

   (b) Find $f(3,\ 4,\ 5)$.

5. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map.

   (a) Can $\ker(f)$ be 1-dimensional and $\text{range}(f)$ be 1-dimensional? If yes, give an example for such f. If not, explain why.

   (b) Can $\ker(f)$ be 1-dimensional and $\text{range}(f)$ be 2-dimensional? If yes, give an example for such f. If not, explain why.

   Hint: Use rank-nullity theorem.

6. Consider the following vectors:

$$\begin{aligned} v_1 &= (1, 2, 3) \\ v_2 &= (-1, 3, -1) \\ v_3 &= (0, 2, 1) \end{aligned}$$

   (a) Check if v_1, v_2, v_3 form a basis for $\mathbb{R}^3$.

(b) Put $S = \{v_1, v_2, v_3\}$. Find the coordinates of vector $v = (2, 1, 0)$ with respect to basis S.
In other words, find c_1, c_2, c_3 such that $v = c_1 v_1 + c_2 v_2 + c_3 v_3$.
Note: one also denotes $[v]_S = (c_1, c_2, c_3)$.