Solutions to some problems in Homework 8:

1)

$$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -6 & -2 \\ -8 & 9 & 3 \end{bmatrix}$$

(b) $$[a]_S = P^{-1}a = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -6 & -2 \\ -8 & 9 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ -13 \\ 19 \end{bmatrix}$$

(c) $$f(a) = f(1,2,3) = (3,-4,4)$$

$$[f(a)]_S = P^{-1}f(a) = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -6 & -2 \\ -8 & 9 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix} = \dots$$

(d) The matrix representing f in standard basis is

$$A = \begin{bmatrix} | & | & | \\ f(e_1) & f(e_2) & f(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{tikzcd} x \arrow[r, "A"] & y \arrow[d, "P^{-1}"] \\ {[x]_S} \arrow[u, "P"] \arrow[r, "{[f]_S}"] & {[y]_S} \end{tikzcd}$$

$$[f]_S = P^{-1}AP$$

$$= \begin{bmatrix} 4 & -5 & -2 \\ 5 & -6 & -2 \\ -8 & 9 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{bmatrix}$$

$$= \dots$$

5)

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & -1 \\ 1 & 1-\lambda \\ 0 & 0 \end{matrix}$$

$$= (1-\lambda)\lambda^2 + (1-\lambda)$$

$$= (1-\lambda)(\underbrace{1+\lambda^2}_{\text{two roots } \lambda = \pm i})$$

There are 3 eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = i, \quad \lambda_3 = -i$$

Find the eigenvectors of λ_1:

$$A - I = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

nonpivot col.

$$x_3 = t, \quad x_2 = 0, \quad x_1 = 0$$

$$E(\lambda_1) = \left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find the eigenvectors of λ_2:

$$A - iI = \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$$

$$\xrightarrow[R_2 = R_2 - R_1]{R_1 = R_1/(-i)} \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-i \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{R_3 = R_3/(1-i)} \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

nonpivot col.

$x_2 = t$

$x_3 = 0$

$x_1 = it$

$$E(\lambda_2) = \left\{ \begin{bmatrix} it \\ t \\ 0 \end{bmatrix} : t \in \mathbb{C} \right\}$$

$$= \text{span}\left\{\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}\right\}$$

Since $\lambda_3 = \overline{\lambda_2}$ (complex conjugate), the eigenvectors of λ_3 are the complex conjugate of the eigenvectors of λ_2.

$$E(\lambda_3) = \text{span}\left\{\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}\right\}$$