Solution to some problems in HW 9:

1 d)

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 1 \\ 0 & -5 & 4 \end{bmatrix}$$

* Find eigenvalues:

$$\det(A - \lambda I_3) = \left|\begin{array}{ccc|cc} 1-\lambda & 4 & -2 & 1-\lambda & 4 \\ 0 & -\lambda & 1 & 0 & -\lambda \\ 0 & -5 & 4-\lambda & 0 & -5 \end{array}\right.$$

$$= \cdots$$

$$= (1-\lambda)(\lambda^2 - 4\lambda + 5)$$

Three distinct roots : $\lambda_1 = 1$, $\lambda_2 = 2+i$, $\lambda_3 = 2-i$

Thus, A is diagonalizable.

* Eigenspace of λ_1:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 4 & -2 \\ 0 & -1 & 1 \\ 0 & -5 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

non-pivot column

$$\begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$E(\lambda_1) = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} \right\}$$

* Eigenspace of λ_2:

$$A - \lambda_2 I = \begin{bmatrix} -1-i & 4 & -2 \\ 0 & -2-i & 1 \\ 0 & -5 & 2-i \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 = R_2/(-2-i) \\ R_3 = R_3 + 5R_2}]{R_1 = R_1/(-1-i)} \begin{bmatrix} 1 & -2+2i & 1-i \\ 0 & 1 & \frac{-2+i}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - (-2+2i)R_2} \begin{bmatrix} 1 & 0 & \frac{3+i}{5} \\ 0 & 1 & \frac{-2+i}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

← non pivot column

$x_3 = t$

$x_2 = -\frac{-2+i}{5}t = \frac{2-i}{5}t$

$x_1 = -\frac{3+i}{5}t$

$$E(\lambda_2) = \left\{ \begin{bmatrix} \frac{-3-i}{5}t \\ \frac{2-i}{5}t \\ t \end{bmatrix} : t \in \mathbb{C} \right\} = \text{span}\left\{ \underbrace{\begin{bmatrix} -3-i \\ 2-i \\ 5 \end{bmatrix}}_{v_2} \right\}$$

* Find eigenspace of λ_3:

Because $\lambda_3 = \overline{\lambda_2}$ (complex conjugage),

$$E(\lambda_3) = \text{span}\{\overline{v_2}\} = \text{span}\left\{ \underbrace{\begin{bmatrix} -3+i \\ 2+i \\ 5 \end{bmatrix}}_{v_3} \right\}$$

Then

$$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -3-i & -3+i \\ 0 & 2-i & 2+i \\ 0 & 5 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$$

$D = P^{-1}AP$

3) $$\left.\begin{aligned} x_{n+1} &= 2x_n + 3x_{n-1} \\ x_n &= x_n \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}}_{y_n} = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}}_{y_{n-1}}$$

$$y_n = A y_{n-1} = A^2 y_{n-2} = \cdots = A^n y_0$$

To find A^n, we diagonalize A:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 3$

* Eigenspace of λ_1:

$$A - \lambda_1 I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

nonpivot column

$x_2 = t$

$x_1 = -t$

$$E(\lambda_1) = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

v_1

* Find eigenspace of λ_2:

$$A - \lambda_2 I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

nonpivot col.

$x_2 = t$

$x_1 = 3t$

$$E(\lambda_2) = \left\{ \begin{bmatrix} 3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

v_2

$$P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A = PDP^{-1}.$$

$$P^{-1} = -\frac{1}{4}\begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$A^n = P D^n P^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 3^n \end{bmatrix}\frac{1}{4}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(-1)^n & 3^{n+1} \\ (-1)^n & 3^n \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (-1)^n + 3^{n+1} & -3(-1)^n + 3^{n+1} \\ -(-1)^n + 3^n & 3(-1)^n + 3^n \end{bmatrix}$$

Then

$$y_n = A^n y_0 = \frac{1}{4}\begin{bmatrix} (-1)^n + 3^{n+1} & -3(-1)^n + 3^{n+1} \\ -(-1)^n + 3^n & 3(-1)^n + 3^n \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (-1)^n + 3^{n+1} \\ -(-1)^n + 3^n \end{bmatrix}$$

Since $y_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$,

$$x_n = \frac{-(-1)^n + 3^n}{4}$$