

Lab 2  
Due 10/19/2018

## I Instruction.

In this lab, we will use Matlab to

- (a) Find the inverse of a matrix by Gauss–Jordan elimination.
- (b) Solve systems of linear equations by Gauss–Jordan elimination or matrix inverse.
- (c) Visualize the linear transformation associate with a matrix.
- (d) Calculate determinant and apply Cramer’s rule to solve systems of linear equations.
- (e) Check linear dependency of given vectors.

The first part is for practice. You don’t need to include this part in your report. The second part (four exercises) is what you write a report on. Please review Lab 1 if you forget how to input a matrix or change decimal format to fraction. **You will need to explain briefly each step, not just show Matlab code.** A good length for this report is 3 - 4 pages.

## II Practice.

**Practice 1:** Find the inverse of a matrix by Gauss–Jordan elimination.

Consider the matrix

$$A = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 4 & 5 \\ 1 & 1 & 3 \end{bmatrix}$$

To find the inverse of  $A$  by Gauss–Jordan elimination method, we attach the identity matrix  $I_3$  on the right hand side of  $A$ . The joint matrix is entered in Matlab by the command

```
>> B = [A eye(3)]
```

which gives

$$B = \left[ \begin{array}{ccc|ccc} 0 & -3 & -1 & 1 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

The vertical bar won’t show up in the output, but let’s imagine that it is there. Now you can perform row operations on  $B$  to transform it into reduced row echelon form (RREF):  $R_1 \leftrightarrow R_2$ ,  $R_3 = R_3 - R_1, \dots$  We already did this in Lab 1 (Practice 1). Note that since  $B$  changes during the elimination process, you may want to store the original matrix before you start the process, say

```
>> C = B
```

If the matrix on the left of the bar in the RREF of  $B$  is exactly the identity matrix then  $A$  is invertible. In this case, the RREF of  $B$  is

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -7/3 & -8/3 & 11/3 \\ 0 & 1 & 0 & -2/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right]$$

The matrix on the right of the bar is the inverse of  $A$ . To extract it from  $B$ , you can use the command

```
>> Ainv = B(:, 4:6) % taking all rows, columns from 4 to 6
```

Matlab also provides a direct function to compute the inverse of the matrix:

```
>> inv(A)
```

Now use the command *inv* to compute the inverse of

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What do you get?

**Practice 2:** Use matrix inverse to solve systems of linear equations.

Consider a system of 3 equations and 3 unknowns:

$$\begin{cases} x + 2y = 5 \\ y - 3z = 5 \\ 3x - z = 4 \end{cases}$$

So far, we have known at least two ways to solve the system. The first method is to use Gauss–Jordan elimination on the augmented matrix. The second is to use inverse of the coefficient matrix. Note that the second method is applicable *only if* the coefficient matrix is invertible, while the first method is always applicable. For both methods, we first write the system in matrix form:  $AX = b$ , where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}.$$

Let's consider *the first method*. The augmented matrix is obtained by attaching column  $b$  on the right hand side of the coefficient matrix  $A$ :

```
>> Aug = [A b]
```

We then perform row operations on the augmented matrix to bring it to RREF as in Practice 1, or use a direct command of Matlab:

```
>> rref(Aug)
```

which gives

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

From here we see that the system has a unique solution  $x = 1$ ,  $y = 2$ ,  $z = -1$ . *The second method* to solve the system is as follows. First, we check if the coefficient matrix is invertible:

```
>> Ainv = inv(A)
```

In this example, Matlab returns a matrix without an error message. This means  $A$  is invertible. Then the solution is  $X = A^{-1}b$ , entered in Matlab by the commands:

```
>> output = num2cell(Ainv*b)
```

```
>> [x,y,z] = output{:}
```

If the coefficient matrix is not invertible, we can only use the first method. Consider such an example:

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ 3x + 3y + 3z = 3 \end{cases}$$

By the command `inv`, you see that the coefficient matrix is not invertible. The RREF of augmented matrix suggests that the system has infinitely many solutions. The free variables are  $y = s$  and  $z = t$ . We get  $x = 3 - s - t$ . The solutions can be written in *parametric form* as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

**Practice 3.** Visualize the linear transformation associate with a matrix.

Consider a linear transformation on the plane  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $f(x_1, x_2) = (x_1 - x_2, 2x_1 + 3x_2)$ , which can be written in matrix form as:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matrix  $A$  is a matrix representing  $f$ . You can draw various shapes and graphs with the `plot` command. This command essentially takes in a list of points and produces a picture by connecting every pair of consecutive points by line segments. The basic syntax is: **plot(x,y)**. Here  $x$  is the list of first coordinates, and  $y$  is the list of second coordinates. The lengths of  $x$  and  $y$  have to be the same. For example, to plot a triangle with vertices at  $(0,0)$ ,  $(2,1)$ ,  $(1,2)$ , we only need to draw a list of 4 points  $(0,0)$ ,  $(2,1)$ ,  $(1,2)$ ,  $(0,0)$ . Specifically, we enter the two vectors of first and second coordinates:

```
>> x1 = [0 2 1 0];
>> x2 = [0 1 2 0];
>> x = [x1;x2];    % for later use
>> plot(x1,x2)
```

Don't close the plot window. Side note: If you want to mark the vertices, use:

```
>> plot(x1,x2,'-o')
```

If you want to explore other options of the `plot` command, type:

```
>> help plot
```

Now  $x$  is a matrix of size  $2 \times 4$ . Each column represents a vertex of the square. Thus,  $x$  encodes all information about the triangle. To find image of each vertex under the map  $f$ , we simply use multiplication by matrix  $A$ :

```
>> y = A*x;
```

The plot of  $y$  gives us the image of the square under  $f$ . Suppose we want to draw  $x$  and  $y$  on the same window for convenience:

```
>> hold on      % to draw on the same plot window
>> axis equal   % the length scales on both axes to be equal
>> plot(y(1,:), y(2,:))
```

You probably already noticed that it is hard to keep track of so many commands on the Matlab window. A way to get around this inconvenience is to create a script file (.m) by pressing `Ctrl + N`. You can put all commands in this section in one file. Then press `F5` to run everything at once.

Suppose we want to draw the parabola  $x_2 = x_1^2$  (say, over the interval  $-1 \leq x_1 \leq 1$ ) together with its image under  $f$ . With the procedure as above, we see that the problem boils down to draw a parabola. The parabola given by parametric equation  $(x_1, x_2) = (t, t^2)$ . We will need a very fine list of points on the parabola. Each point has coordinate  $(t, t^2)$ .

```
>> x1 = -1:0.01:1;    % sample the interval [-1,1] with step size 0.01
>> x2 = x1.^2;
```

The dot is to indicate that we want to square each component of the vector, not the vector itself. You can proceed at this point to draw the image of the parabola under  $f$ .

**Practice 4:** Determinant and Cramer's rule.

Consider the following system of equations:

$$\begin{cases} -3x_2 - x_3 & = & 1 \\ x_1 + 4x_2 + 5x_3 & = & 0 \\ x_1 + x_2 + 3x_3 & = & 2 \end{cases}$$

The matrix form is  $AX = b$  with

$$A = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 4 & 5 \\ 1 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

To check if  $A$  is invertible, compute its determinant:

```
>> det(A)
```

Now that  $A$  is invertible, we can solve the given system by Cramer's rule. The solution is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad x_3 = \frac{|A_3|}{|A|},$$

where  $A_i$  is the matrix obtained from  $A$  by replacing the  $i$ 'th column of  $A$  by  $b$ . For example,

```
>> A2 = A; A2(:, 2) = b
```

You can proceed at this point to compute  $x_1, x_2, x_3$ .

**Practice 5:** Check linear dependency of given vectors.

Consider three vectors  $v_1 = (1, 0, 2, 3)$ ,  $v_2 = (2, -1, -1, 2)$ ,  $v_3 = (0, 1, 5, 4)$ . Suppose we want to check if they are linearly dependent. There are several ways to do it. A natural way is to use the definition of linear dependency: consider the equation

$$c_1v_1 + c_2v_2 + c_3v_3 = 0. \tag{1}$$

This is actually a system of 4 equations and 3 unknowns  $c_1, c_2, c_3$ . If the system has only one solution (the trivial solution), then  $v_1, v_2, v_3$  are linearly independent. Otherwise, they are linearly dependent. You can realize that the coefficient matrix of this system is made of  $v_1, v_2, v_3$  arranged as columns.

```
>> v1 = [1;0;2;3]
>> v2 = [2;-1;-1;2]
>> v3 = [0;1;5;4]
>> A = [v1 v2 v3]
```

Then you solve for  $c_1, c_2, c_3$  as in Practice 2. In the RREF form, the third column is nonpivot. You conclude at this point that the system does not have a unique solution, and therefore, the given vectors are linearly dependent. We know that one vector must be a linear combination of the other

two. How to find such a relation? To answer this question, we need to solve the system. First,  $c_3$  is a free variable. Then

$$c_3 = t, \quad c_2 = t, \quad c_1 = -2t.$$

Fix an arbitrary value of  $t$ , say  $t = 1$ . Then substitute  $c_1, c_2, c_3$  into Equation (1). We get  $-2v_1 + v_2 + v_3 = 0$ . Therefore,  $v_3 = 2v_1 - v_2$ .

### III Exercises.

1. Compute the inverse of the following matrix by Gauss-Jordan elimination (as in Practice 1). Make sure that you show every row operation (as in Exercise 1 of Lab 1). Then double check your result with commands *rref* and *inv*.

$$A = \begin{bmatrix} 3 & 1 & 10 \\ 2 & 5 & 4 \\ -1 & -3 & 1 \end{bmatrix}$$

2. Solve the following systems of equations using the augmented matrix method, or inverse matrix method, or Cramer's rule. Make sure that each method is used at least once. If a system has infinitely many solutions, express the solutions in parametric form as in Practice 2. You can use the commands *rref* and *inv*.

$$\begin{array}{ll} \text{(a)} & \begin{cases} x_1 + 2x_2 + 3x_3 & = 1 \\ -x_1 + 7x_3 + 5x_4 & = 0 \\ 2x_1 + x_2 + 2x_3 + 3x_4 & = 2 \\ 4x_1 + 2x_2 + x_3 + 2x_4 & = 1 \end{cases} \\ \text{(c)} & \begin{cases} x_1 - 3x_2 + 2x_3 & = 3 \\ -x_1 + x_2 - x_3 + x_4 & = -2 \\ 3x_1 - x_2 + x_3 - x_4 & = -2 \\ -2x_1 - x_2 + 2x_4 & = 1 \end{cases} \end{array}$$

$$\begin{array}{ll} \text{(b)} & \begin{cases} 2x_1 - x_3 + 3x_4 & = 2 \\ x_1 + x_2 + 2x_3 & = 1 \\ 3x_1 - x_2 + 3x_4 & = -1 \\ 7x_1 + x_2 + 3x_3 + 6x_4 & = 4 \end{cases} \\ \text{(d)} & \begin{cases} 2x_1 + 3x_2 + 11x_3 + 5x_4 & = 2 \\ x_1 + x_2 + 5x_3 + 2x_4 & = 1 \\ 2x_1 + x_2 - 3x_3 + 2x_4 & = -3 \\ x_1 + x_2 - 3x_3 + 4x_4 & = -3 \end{cases} \end{array}$$

3. Consider a linear transformation on the plane  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $f(x_1, x_2) = (x_1 + 2x_2, -x_1 + x_2)$ .
  - (a) Draw the image under  $f$  of the square with vertices at  $(0,0)$ ,  $(1,0)$ ,  $(1,1)$ ,  $(0,1)$ . What does it look like? What is its area?
  - (b) Draw the image under  $f$  of the unit circle  $x_1^2 + x_2^2 = 1$ . What does it look like? Hint: the unit circle is given by parametric equation  $(x_1, x_2) = (\cos t, \sin t)$  with  $0 \leq t \leq 2\pi$ .
4. Check if the following vectors are linear dependent. If they are, express one vector as the linear combination of the other three.

$$\begin{aligned} v_1 &= (0, 2, 0, -3) \\ v_2 &= (1, 3, 5, 7) \\ v_3 &= (-2, -6, -2, 0) \\ v_4 &= (1, 2, -1, -2) \end{aligned}$$