Lab 4

Due 11/23/2018

I Instruction.

In this lab, we will use Matlab to

(a) Find coordinates of vectors with respect to a basis.

(b) Find the matrix representing a linear map with respect to a basis.

(c) Learn some simple applications of linear algebra in geometry and ecology.

Important commands and outputs should be shown. You also need to explain briefly each step, not only show Matlab code. You are allowed to directly use the commands *rref* and *inv* if need to.

II Practice.

Practice 1:

Consider the vectors

$$\begin{aligned} v_1 &= (2,\, 6,\, 5) \\ v_2 &= (5,\, 3,\, -2) \\ v_3 &= (7,\, 4,\, -3) \end{aligned}$$

To check if these vectors form a basis for $\mathbb{R}^3$, we only need to check if they are linearly independent (since we already have the right number of vectors). To do so, we form a matrix whose columns are made of these vectors:

```
>> P = [v1 v2 v3]
```

Then check if determinant of A is nonzero:

```
>> det(P)
```

Note that if the determinant computed by Matlab is too close to zero, say 10^{-17} for example, then the true determinant should be zero and therefore the matrix is singular. The very small error is due to round-off or truncation error in Matlab. Think of the basketweave method: there is no reason why a matrix made of nice whole-number entries could end up having such a strange determinant. (This was the case in some matrices in Exercise 2 of Lab 2.) But in our case, the determinant is equal to -1. Thus, $S = \{v_1,\, v_2,\, v_3\}$ is indeed a basis of $\mathbb{R}^3$.

Let $b = (2, 3, 1)$. This triple is the coordinate of b in the standard basis $S_0 = \{e_1,\, e_2,\, e_3\}$. As a convention, a vector is identified with its coordinate in the standard basis. In other words, we consider b (a vector or a point in space) as $(2, 3, 1)$, the coordinate of b in standard basis S_0. One can write b interchagebly with $[b]_{S_0}$. Recall that the coordinate of b in basis S is a triple $(c_1,\, c_2,\, c_3)$ such that $b = c_1v_1 + c_2v_2 + c_3v_3$. In matrix form,

$$b = P \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = P[b]_S$$

Thus, $[b]_S = P^{-1}b$. In Matlab,

```
>> bS = inv(P)*b
```

Practice 2:
A linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniquely defined if we know what f does on a basis of $\mathbb{R}^n$. Let us consider an example. The vectors

$$\begin{aligned} v_1 &= (2,\, 6,\, 5) \\ v_2 &= (5,\, 3,\, -2) \\ v_3 &= (7,\, 4,\, -3) \end{aligned}$$

form a basis S of $\mathbb{R}^3$ as discussed previously. A linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ would be well-defined and unique if we know $f(v_1)$, $f(v_2)$, $f(v_3)$. Suppose

$$\begin{aligned} f(v_1) &= (1,\, 2) = w_1 \\ f(v_2) &= (3,\, -1) = w_2 \\ f(v_3) &= (0,\, -4) = w_3 \end{aligned}$$

Any vector $b \in \mathbb{R}^3$ is a linear combination of v_1, v_2, v_3. Write $b = c_1v_1 + c_2v_2 + c_3v_3$. We can compute $f(b)$ as follows:

$$\begin{aligned} f(b) &= c_1f(v_1) + c_2f(v_2) + c_3f(v_3) \\ &= \underbrace{\begin{bmatrix} | & | & | \\ f(v_1) & f(v_2) & f(v_3) \\ | & | & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[b]_S} \\ &= QP^{-1}b. \end{aligned}$$

From here we see that the matrix representing f in the standard basis is $A = QP^{-1}$. The implementation in Matlab is rather simple:

```
>> Q = [w1 w2 w3]
>> A = Q*inv(P)
```

To find $f(4, 2, 3)$ for example, we use the fact that $f(b) = Ab$:

```
>> b = [4;2;3]
>> fb = A*b
```

Practice 3:
Consider a linear map $g : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$\begin{aligned} g(v_1) &= v_1 - v_3, \\ g(v_2) &= 2v_1 + v_2, \\ g(v_3) &= v_2 + v_3. \end{aligned}$$

where v_1, v_2, v_3 are given in Practice 1. Suppose we want to find an explicit formula of g in standard basis. From the description of g, we see that the matrix representing g in basis S is:

$$[g]_S = \begin{bmatrix} | & | & | \\ [g(v_1)]_S & [g(v_2)]_S & [g(v_3)]_S \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Denote by A the matrix representing g in the standard basis, which is the matrix we want to find. We know that A and $[g]_S$ are related to each other by $[g]_S = P^{-1}AP$. Multiplying both sides by

P on the left, and by P^{-1} on the right, we get $A = P[g]_S P^{-1}$. The result is

$$A = \begin{bmatrix} -23 & 26 & -23 \\ -379 & 410 & -340 \\ -431 & 465 & -384 \end{bmatrix}$$

Thus, the explicit formula of g in standard basis is

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -23 & 26 & -23 \\ -379 & 410 & -340 \\ -431 & 465 & -384 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -23x + 26y - 23z \\ -379x + 410y - 340z \\ -431x + 465y - 384z \end{bmatrix}$$

Practice 4:

Let us consider some applications of linear algebra in geometry. Almost all commonly-used geometry transformations are linear maps, except for the translations (simply because a translation does not map the origin to the origin).

Consider the plane (P): $x + 2y - z = 0$ in the space. The (perpendicular) projection onto (P) maps each point in the space to its projection on (P). How to write an explicit formula for this linear map? We call it $f : \mathbb{R}^3 \to \mathbb{R}^3$. To identify f, we only need to know what f does on a basis of $\mathbb{R}^3$. We would like to choose three special vectors in $\mathbb{R}^3$ that have simple projections onto (P). Any vector on (P) is mapped to itself under f. Two linearly independent directions in (P) are $v_1 = (1, 0, 1)$ and $v_2 = (1, 1, 3)$. We know that $f(v_1) = v_1$ and $f(v_2) = v_2$. Another special direction is the direction perpendicular to the plane (P). A vector in this direction can be seen from the equation of (P), which is $v_3 = (1, 2, -1)$. We know that f maps v_3 to the origin. In other words, $f(v_3) = 0$. Now that we have

$$\begin{aligned} f(v_1) &= v_1, \\ f(v_2) &= v_2, \\ f(v_3) &= 0. \end{aligned}$$

Technically, at this point f is uniquely defined. To make it more explicit, we can find an explicit formula for f in the standard basis using the method in Practice 3. We also see that v_1, v_2, v_3 are eigenvectors of f corresponding to eigenvalues 1, 1, 0 respectively. (Recall that an eigenvector is a direction along which f acts by scaling. The corresponding eigenvalue is the scaling factor on that direction.)

Another useful geometric transformation is the rotations. It is a little involved to describe rotations in 3-dimensional space, so let us consider rotations on the 2-dimensional plane instead. Let θ be the angle of rotation. The rotation is a linear map $g : \mathbb{R}^2 \to \mathbb{R}^2$. To identify g, we need to know how g acts on a basis of $\mathbb{R}^2$. It does not seem clear what directions are special. We will consider the vectors in the standard basis: $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The ending points of these vectors lie on the unit circle. The image of e_1 under the rotation is a point on the unit circle at angle θ with respect to the x-axis. This point has coordinate $(\cos\theta, \sin\theta)$. The image of e_2 under the rotation is a point on the unit circle at angle $\theta + \frac{\pi}{2}$ with respect to the x-axis. This point has coordinate $(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) = (-\sin\theta, \cos\theta)$. We have found that

$$\begin{aligned} g(e_1) &= (\cos\theta, \sin\theta), \\ g(e_2) &= (-\sin\theta, \cos\theta) \end{aligned}$$

The matrix representing g in the standard basis is

$$A = \begin{bmatrix} | & | \\ g(e_1) & g(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The explicit formula of g in standard basis is therefore

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

If $\theta = 0$ (or $\theta = k2\pi$ for some integer k), the rotation is simply the identity map. Every direction is preserved. In other words, every vector is an eigenvector. The eigenvalue (scaling factor) is 1. If θ is otherwise, the rotation does not preserve any directions and thus has no real-valued eigenvectors. However, it has complex-valued eigenvectors as well as complex-valued eigenvalues.

Practice 5:

To compute the eigenvectors of a square matrix (or a linear map), we usually start with computing the eigenvalues by solving the equation

$$\det(A - \lambda I_n) = 0.$$

This is the problem of finding roots of a polynomial. The degree of the polynomial is equal to the size of matrix A. If A has size greater than 4, we run into trouble. In fact, it was proven by Ruffini and Abel around 1799–1825 that a general polynomial of degree 5 or higher cannot be solved by radicals. The strategy is to avoid solving for *all* eigenvalues before finding the eigenvectors. There exist many numerical method to do so. Let us consider a simple method called *power iteration*. This method only produces the largest (in sense of absolute value) eigenvalue and a corresponding eigenvector. Such an eigenvalue is called *dominant eigenvalue*. The direction of the corresponding eigenvector is called a *principal direction*. This direction is important in many applications. The power iteration method only works under certain conditions:

- The matrix has n distict eigenvalues.
- The absolute values of the eigenvalues are also distinct.

If you pick randomly n^2 real numbers to form an $n \times n$ matrix, almost surely the resulting matrix will satisfy the above conditions. Therefore, the conditions are almost always satisfied. Let us consider an example:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

To find a principal direction of A, we pick some direction to start with, say $v_0 = (2, 3)$. Then update the direction by the following procedure:

- The updated direction is Av_0. The normalized direction is vector $v_1 = Av_0/\|Av_0\|$ where $\|Av_0\|$ is the magnitude of vector Av_0.
- The updated direction is Av_1. The normalized direction is vector $v_2 = Av_1/\|Av_1\|$.
- (continue this procedure)

The more steps we make, the closer the resulting direction (v_m after m steps) is to the principal direction (an eigenvector corresponding to the largest eigenvalue). The reason why we normalize the direction is to prevent the magnitude of the resulting vector from growing too large. Imagine that after a certain number of steps the resulting direction seems to settle on a direction v. Then v is a fixed point in this loop, i.e.

$$v = \frac{Av}{\|Av\|}$$

If we denote $\lambda = \|Av\|$ then $Av = \lambda v$. Then λ is an eigenvalue (it is in fact the dominant eigenvalue), and v is a corresponding eigenvector. There is a subtle issue worth noting: for the procedure to converge, the dominant eigenvalue must be positive. This can be overcome by checking if the quantity $\frac{(Av_m)\cdot v_m}{v_m \cdot v_m}$ converges (instead of checking if v_m converges). However, we will not focus on this issue here. The recursive procedure above can be programed in Matlab as a function:

```
function v = powerIter(A,v0,m)
v = v0
for i = 1:m
    v = A*v/norm(A*v)
end
```

The inputs are: matrix A, the initial direction v_0 (which is a column vector) to start with, and the number of steps m. The output is the resulting direction at the m'th step. (The resulting directions at all previous steps are also printed out.) Copy the above code segment to a new script file (.m) and save under the name powerIter.m (Make sure that the file's name is the same as the function's name, which is "powerIter" in this case.) Now apply this function for $m = 30$ for example.

```
>> v = powerIter(A,v0,30)
```

Since what step do you notice that the resulting direction starts to converge? The limiting direction seems to be $v = (0.7071,\ 0.7071)$. This is an eigenvector corresponding to the dominant eigenvalue:

```
>> lambda = norm(A*v)
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which gives $\lambda = 4$. Now try using a different initial direction, say $v_0 = (50,\ 39)$. Will the limiting direction and dominant eigenvalue change?

Another way to look at the above procedure is that v_m, the resulting direction at the m'th step, is actually given by

$$v_m = \frac{A^m v_0}{\|A^m v_0\|}.$$

You can verify by hand this formula for small values of m without difficulty. This is why the method has the name "power iteration". If v_m converges to v as $m \to \infty$, then $A^m v_0$ tends to be aligned with (i.e. parallel to) v, the principal direction of A, for large m regardless of the choice of v_0.

III Exercises.

1. Consider the vectors

$$\begin{aligned} v_1 &= (3,\ 7,\ 0,\ 0) \\ v_2 &= (2,\ 5,\ 0,\ 0) \\ v_3 &= (1,\ 2,\ 9,\ 11) \\ v_4 &= (2,\ 5,\ 4,\ 5) \end{aligned}$$

Check if they form a basis of $\mathbb{R}^4$. Then find the coordinate of the vectors $a = (2,\ 4,\ -1,\ 3)$ and $b = (1,\ -3,\ 5,\ 2)$ in basis $S = \{v_1,\ v_2,\ v_3,\ v_4\}$.

2. Let v_1, v_2, v_3, v_4 be given as in the previous exercise. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ and $g : \mathbb{R}^4 \to \mathbb{R}^4$ be linear maps satisfying:

$$\begin{aligned} f(v_1) &= (1,\, 3,\, 4) \\ f(v_2) &= (-1,\, 0,\, 3) \\ f(v_3) &= (0,\, 5,\, 2) \\ f(v_4) &= (-2,\, -3,\, 1) \end{aligned}$$

and

$$\begin{aligned} g(v_1) &= v_1 + v_2 - 2v_4 \\ g(v_2) &= 2v_1 + 3v_2 - 4v_3 \\ g(v_3) &= 4v_1 - 2v_2 - v_3 + v_4 \\ g(v_4) &= 2v_4 \end{aligned}$$

   (a) Find the matrices representing f in the standard basis.
   (b) Find the matrices representing g in basis $S = \{v_1,\, v_2,\, v_3,\, v_4\}$ and in the standard basis.
   (c) Find the matrix representing the composite map $f \circ g$ in the standard basis.

3. On the 2-dimensional plane, consider the mirror reflection about the line

$$(l) : \; x + 2y = 0$$

   (a) Form a matrix representing the transformation in the standard basis.
   (b) Determine the eigenvectors and eigenvalues of this transformation.
   (c) Determine the image of the point $(-20,\, 39)$ under the transformation.

4. On the 2-dimensional plane, consider a transformation composed of a rotation about the origin by $30°$ (which is $\pi/6$ radian) followed by a scaling by factor 2.

   (a) Form a matrix representing the transformation in the standard basis.
   (b) Determine the real-valued eigenvectors and real-valued eigenvalues (if exist) of this transformation.
   (c) Determine the image of the point $(-1,\, 3)$ under the transformation.

5. Consider an ecosystem consisting of lions, hyenas and antelopes. Each species is counted at the end of each year. To give an approximate mathematical model for the system, we adopt the convention that the population (i.e. the number of individuals) is not necessarily a whole number. Denote by x_n, y_n, z_n the population of lions, hyenas and antelopes respectively counted at the end of year n. Over the years, suppose ecologists observe that:

   - The population of lions is equal to 101% of their population of the previous year (due to minimal reproduction rate), minus 1% of the hyena population of the previous year (due to food competition and fightings), plus 3% of the antelope population of the previous year (due to food supply).
   - The population of hyenas is equal to 102% of their population of the previous year, minus 1% of the lion population of the previous year, plus 5% of the antelope population of the previous year.
   - The population of antelopes is equal to 140% of their population of the previous year, minus 10% of the lion population of the previous year, minus 10% of the hyena population of the previous year.

(a) Express a relation between x_{n+1}, y_{n+1}, z_{n+1} and x_n, y_n, z_n in matrix form.

(b) Suppose the initial populations ($n = 0$) are 7 lions, 15 hyenas and 100 antelopes. Determine the population of each species at the end of the sixth year.

(c) Determine the ratio of lion : hyena : antelope at an ecosystem equilibrium (i.e. the limiting ratio over the years). Does it change when you vary the initial populations x_0, y_0, z_0?