Lecture 10 (10/12/2018)

A invertible is equivalent to

- $A \sim I_n$
- $\det A$ (also denoted by $|A|$) $\neq 0$.
- $\operatorname{rank}(A) = n$ (A is full-rank)

Determinant of special matrices:

(1)
$$\begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

By a chain of row operations of type $R_i = R_i - cR_j$, one can transform A into identity matrix I_n.

$|A| = |I_n| = 1$

(2)
$$\begin{bmatrix} d_1 & & & * \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

Suppose $d_1, \dots, d_n \neq 0$.

$$|A| \overset{R_1 = \frac{1}{d_1}R_1}{=\!=} d_1 \begin{vmatrix} 1 & & & * \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{vmatrix} \overset{R_2 = \frac{1}{d_2}R_2}{=\!=} d_1 d_2 \begin{vmatrix} 1 & & & & * \\ & 1 & & & \\ & & d_3 & & \\ & & & \ddots & \\ 0 & & & & d_n \end{vmatrix} = \dots$$

$$= d_1 d_2 \dots d_n$$

If one of $d_1, \dots, d_n$ is equal to 0, the matrix is not full rank (because there will be at least one nonpivot column in RREF form).

In this case, $|A| = 0$.

In both cases, $|A| = d_1 d_2 \cdots d_n$.

$$\boxed{\text{The determinant of an upper triangular matrix is equal to the product of entries on the diagonal.}}$$

Ex:

$$\begin{vmatrix} 2 & 4 & 6 \\ 2 & 3 & 5 \\ 3 & 1 & 0 \end{vmatrix} \overset{R_2 = R_2 - R_1}{\underset{\substack{R_1 = R_1/2 \\ R_3 = R_3 - 2R_1}}{=\!=}} 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -5 & -15 \end{vmatrix} \overset{R_3 = R_3 - 5R_2}{=\!=} 2 \underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -10 \end{vmatrix}}_{\text{upper triangular}}$$

$= 2 \cdot 1 \cdot (-1)(-10) = 20$

(3) Lower triangular matrices:

$$\begin{vmatrix} d_1 & & 0 \\ & d_2 & \\ * & & \ddots \; d_n \end{vmatrix} = d_1 d_2 \cdots d_n$$

(4) Transpose of a matrix: A^T

$$(A^T)_{ij} = A_{ij}$$

Ex:

$$\underbrace{A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \end{bmatrix}}_{2\times 3} \qquad \underbrace{A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}_{3\times 2}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

What is the relation between $|A|$ and $|A^T|$?

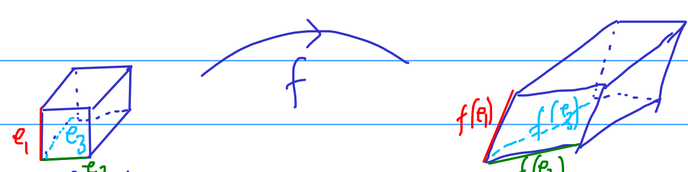
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\det = ad - bc$

For general A, let's assume A ... 3×3 for simplicity.

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \longleftrightarrow f \qquad A^T = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Unit cubic:



vol $= 1$ $\qquad$ vol $= |A| \cdot 1 = |A|$

$\text{vol} = \det \begin{vmatrix} C_1 \\ C_2 \\ C_3 \end{vmatrix} = |A^T|$

$\det(I_n) = 1$

$f(e_1) = f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_1$ .... $f(e_2) = C_2$, $f(e_3) = C_3$

$$\boxed{|A| = |A^T|}$$

Ex:

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix} = 0$$

If A has .... repeated rows
repeated cols.
a zero row
a zero col.
$\Big\}$ then $|A| = 0$.

More applications:

- $AX = 0$ .... homogeneous system

  A ... $n \times n$ matrix

  This system has only trivial sol. if and only if A is invertible. (i.e. $|A| \neq 0$).

- Cramer's rule:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \rightsquigarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Observe:

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} e & b \\ f & d \end{bmatrix}}_{A_1}$$

A_1 obtained from A by replacing the first col. of A by the right hand side.

Take the determinant of both sides

$$|A|\, x = |A_1| \Rightarrow x = \frac{|A_1|}{|A|}$$

Similarly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} = \begin{bmatrix} a & e \\ c & f \end{bmatrix} = A_2$$

Take the determinant of both sides:

$$|A|\, y = |A_2| \Rightarrow y = \frac{|A_2|}{|A|}.$$