Lecture 11 (10/15/2018)

Subspace of $\mathbb{R}^n$ is a subset $V \subset \mathbb{R}^n$ satisfying three following properties:

(a) $0 \in V$

(b) If $v, w \in V$ then $v + w \in V$ (closed under addition).

(c) If $v \in V$ and $c \in \mathbb{R}$ then $cv \in V$ (closed under scalar multiplication).

Examples:

(1) $V = \{0\}$

(2) $V = \{(x_1, x_2) : x_1 = 2x_2\}$ ... line passing through the origin

(3) $V = \{x \in \mathbb{R}^n : Ax = 0\}$ where A is an $m \times n$ matrix

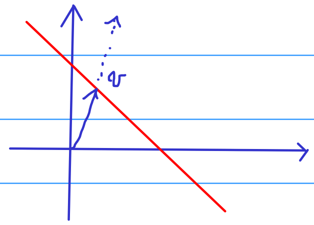
In this case, V is called the **null space** of A.

In the second example,

$$x_1 - 2x_2 = 0 \quad \cdots\cdots \quad \underbrace{\begin{bmatrix} 1 & -2 \end{bmatrix}}_{A \,\cdots\, 1\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = 0$$

Non examples:

(1) $V = \{(x_1, x_2) : x_1 + x_2 = 1\}$ ... line not passing through the origin.

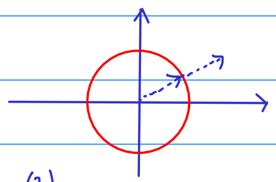


$v = \left(\frac{1}{2}, \frac{1}{2}\right) \in V$

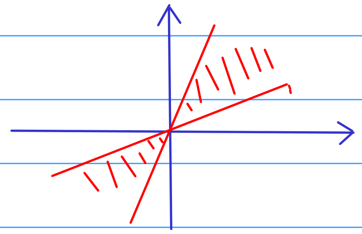
$2v = (1, 1) \notin V$

(2) $V = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$

Circle is a bounded set .... can't be a subspace.



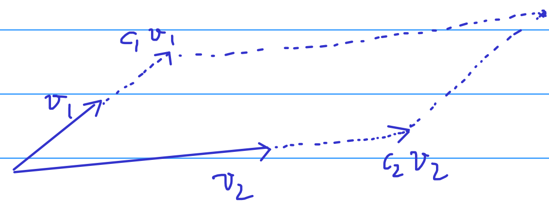
(3)



This subset is closed under scalar multiplication, but not under addition.

(4) $\mathbb{Z} \subset \mathbb{R}^1$ is closed under addition, but not closed under scalar multiplication.

A <u>linear combination</u> of vectors $v_1, v_2, \ldots, v_m$ is a vector of the form $v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$, where $c_1, \ldots, c_m$ are real numbers.

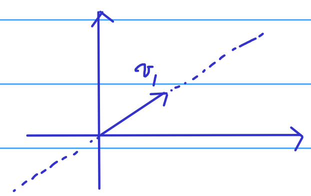


The set V consisting of all linear combinations of vectors $v_1, v_2, \ldots, v_m$ is called the <u>span</u> of $v_1, v_2, \ldots, v_n$. It is the smallest subspace that contains $v_1, \ldots, v_m$. In other words, if W is a subspace containing $v_1, v_2, \ldots, v_m$ then $V \subset W$.

Notation: $\quad V = \text{span}\{v_1, \ldots, v_m\}$.

<u>Example</u>:

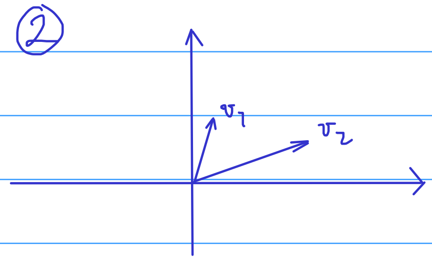
① $v_1 \in \mathbb{R}^2$



$\text{span}\{v_1\} = \{t v_1 : t \in \mathbb{R}\}$

This is a line passing through the origin.

②



$\text{span}\{v_1, v_2\} =$
- $\{0\}$ if $v_1 = v_2 = 0$
- line through the origin if one vector is a multiple of the other.
- plane if v_1 and v_2 are not colinear.

Vectors $v_1, v_2, \ldots, v_m$ are said to be <u>linearly independent</u> if

$$c_1 v_1 + \cdots + c_m v_m \neq 0 \quad \text{unless for } c_1 = \cdots = c_m = 0.$$

In other words, $v_1, \ldots, v_m$ are linearly independent if there are no nontrivial linear relations among them.

Ex:

If v_1, v_2, v_3 satisfy $v_3 = v_1 + v_2$ then they are linearly dependent. A nontrivial linear relation among them is $v_1 + v_2 - v_3 = 0$.