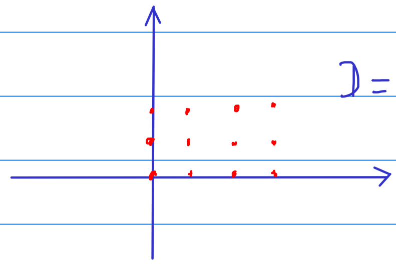
Lecture 12 (10/17/2018)

Review:

V subspace:
- $0 \in V$
- $v, w \in V \Rightarrow v + w \in V$
- $c \in \mathbb{R},\ v \in V \Rightarrow cv \in V$

Subspace of $\mathbb{R}^3$:
- $\{0\}$
- line through the origin
- plane ______
- $\mathbb{R}^3$ itself

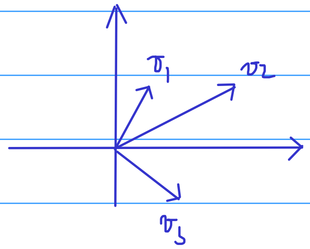


$D = \{(x, y) : x, y \in \mathbb{Z}\}$ is not a subspace. The third condition is violated.

Span$\{v_1, \dots, v_m\}$ consists of all linear combinations of $v_1, \dots, v_m$:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_m v_m$$

The vectors $v_1, \dots, v_m$ are said to be linearly independent if there are no nontrivial linear relations among them: if $c_1 v_1 + \dots + c_m v_m = 0$ then $c_1, \dots, c_m$ must all equal to 0.



* A geometric way to explain linear independence:
Let's consider 3 vectors v_1, v_2, v_3.
- Start at the origin
- walk distance c_1 along direction v_1
- then walk distance c_2 along direction v_2
- then ______ c_3 ______ v_3

These vectors are linearly ind. if you can never return to the origin unless you never move (i.e. when $c_1 = c_2 = c_3 = 0$).

---

* Check if a set of vectors is linearly independent.

Ex:

Consider $v_1 = (1,2,3)$
$v_2 = (0,1,0)$
$v_3 = (2,3,-1)$

The system $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ is equiv. to

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a system of 3 eqs. and 3 unknowns. The matrix form is

$$\begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \\ \downarrow & \downarrow & \downarrow & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right] \end{array}$$

If this sys. has only trivial sol. ($c_1, c_2, c_3 = 0$) then v_1, v_2, v_3 are linearly ind.
Otherwise (when the sys. has inf. many sols.), v_1, v_2, v_3 are linearly dep.

- General rule:

Given vectors $v_1, \dots, v_m$ in $\mathbb{R}^n$. Establish matrix

$$A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_m \\ \vdots & \vdots & & \vdots \end{bmatrix} \quad n \times m \text{ matrix}$$

Then find RREF of A.

If all cols. are pivot, the system $[A \mid 0]$ has only one sol. (the trivial sol.). In this case, $v_1, \dots, v_m$ are linearly independent.

If there is a nonpivot col., there are ∞-many sol. Thus, $v_1, \dots, v_m$ are lin. dep.

Special cases:

- $m > n$: A is a "flat" matrix .... RREF must has a nonpivot column.

  m vectors in $\mathbb{R}^n$ must be linearly dependent.
  (there are too many vectors in $\mathbb{R}^n$ to be lin. ind.)

- $m \leq n$: we have to go through the procedure.

If $m=n$, $\det A = 0$ ..... lin. dep.

$\det A \neq 0$ ..... lin. ind.

Back to the example,

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 0 & -1 \end{vmatrix} = -7 \neq 0.$$ Thus v_1, v_2, v_3 are lin. ind.

* Another problem:

Let $v_1 = (1, 2, 0)$

$v_2 = (2, 1, 1)$

$v = (1, 5, -1)$

Is v a linear comb. of v_1 and v_2?

We want to solve the eq. $v = c_1 v_1 + c_2 v_2$ for c_1, c_2. If this sys. is consistent, then v is indeed a linear comb. of v_1 and v_2.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

Matrix form:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

This sys. is consistent. Solution is $c_1 = 3$, $c_2 = -1$.

$v = 3v_1 - v_2$