Lecture 13 (10/19/2018)

Remark on the last quiz:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies 2A = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} \implies |2A| = 4|A|$$

In general, if A is an $n \times n$ matrix then $|cA| = c^n |A|$.

$$A \xrightarrow[\substack{\cdots \\ R_n = cR_n}]{\substack{R_1 = cR_1 \\ R_2 = cR_2}} cA$$

Last time:

$v_1 = (1,2,1)$

$v_2 = (2,-3,0)$

$v = (7,0,3)$

Is v a linear comb. of v_1 and v_2?

Solve for c_1 and c_2: $c_1 v_1 + c_2 v_2 = v$.

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & -3 & 0 \\ 1 & 0 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

$v = 3v_1 + 2v_2$

Can every vector in $\mathbb{R}^3$ be expressed as a linear comb. of v_1 and v_2?

No!

$$\left[\begin{array}{cc|c} | & | & | \\ v_1 & v_2 & w \\ | & | & | \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & ? \\ 0 & 1 & ? \\ 0 & 0 & ? \end{array}\right]$$

We can always find a vector w such that this number is nonzero, leading to no solutions c_1, c_2.

$$\text{span}\{v_1, v_2\} \subset \mathbb{R}^3$$

and $\text{span}\{v_1, v_2\}$ is strictly smaller than $\mathbb{R}^3$.

There are not enough vectors to span $\mathbb{R}^3$.

<u>Rule</u>:

$\mathbb{R}^n$ cannot be spanned by fewer than n vectors.

Basis: A set $S=\{v_1,\dots,v_m\}$ is said to be a <u>basis</u> of subspace V if:

- $v_1,\dots,v_m$ are linearly independent,
- $\text{span}\{v_1,\dots,v_m\}=V$

Example:

$V=\mathbb{R}^n$

$$\left.\begin{array}{l} e_1=(1,0,\dots,0)\\ e_2=(0,1,\dots,0)\\ \dots\dots\\ e_n=(0,0,\dots,1)\end{array}\right\}\text{ This forms a basis for } \mathbb{R}^n$$

<u>Thm</u>: Every basis of a subspace has the same number of vectors. This number is called the dimension of the subspace.

<u>Ex</u>:

$\mathbb{R}^n$ has dimension n.

<u>Ex</u>:

$V=\mathbb{R}^3$

$v_1=(1,2,3)$ The set $\{v_1,v_2\}$ is not a basis because it has

$v_2=(2,1,2)$ less than 3 vectors

$v_3=(0,3,1)$ The set $\{v_1,v_2,v_3,v_4\}$ is not a basis either.

$v_4=(2,3,4)$ How about $\{v_1,v_2,v_3\}$?

Claim: We only need to check if v_1,v_2,v_3 are lin. ind.

$$\left[\begin{array}{ccc|c} | & | & | & | \\ v_1 & v_2 & v_3 & 0 \\ | & | & | & | \end{array}\right] \text{ has unique sol. ?}$$

$$\det\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}\neq 0 \;\dots\; \text{Yes, they are lin. ind.}$$

Thus, $\{v_1,v_2,v_3\}$ is a basis of $\mathbb{R}^3$.

Procedure:

To check if $\{v_1,\dots,v_m\}$ is a basis for $\mathbb{R}^n$:

- If $m\neq n$, stop. Not a basis
- If $m=n$, check if

$$\det \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \neq 0.$$

Ex: Find basis of the span of vectors.

$v_1 = (1, 0, 2)$

$v_2 = (2, 1, -1)$

$v_3 = (3, 1, 1)$

$v_4 = (5, 2, 0)$

$V = \text{span}\{v_1, v_2, v_3, v_4\}$. What is a basis of V?

There are two methods:

① Remove redundant vectors from the set $S = \{v_1, v_2, v_3, v_4\}$ to make it a basis.

$$\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow$

pivot columns

The pivot columns indicate which vector in S to be kept. In this example, the pivot columns are 1 and 3. Thus, $\{v_1, v_2\}$ is a basis of V.

② Give a basis not made of the original vectors.
(next class)