Lecture 14 (10/22/2018)

Problem: find a basis of the subspace spanned by $v_1, v_2, \dots, v_m \in \mathbb{R}^n$.

There are two methods.

① Idea: remove "redundant" vectors from the set $S = \{v_1, \dots, v_m\}$ to obtain a basis of V.

- Arrange the given vectors as columns of a matrix

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix}$$

(V is spanned by the cols. of A, called the column space of A, denoted by $C(A)$.)

- Reduce A to RREF form, called B.
- The pivot columns of B correspond to the columns of A to be selected. These selected vectors form a basis for V.

Ex

$v_1 = (-1, 2, 1)$

$v_2 = (2, 3, 0)$

$v_3 = (1, 0, 3)$

$V = \text{span}\{v_1, v_2, v_3\}$

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow$ pivot columns

The first and second columns of A are chosen! A basis of V is then $\{v_1, v_2\}$.

Why does this method work?

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 = 3C_1 + 2C_2 \quad \leftarrow \cdots\cdots \quad C_3 = 3C_1 + 2C_2$

Linear relations among columns of B are carried over to cols. of A.

Note that $\dim V = \#$ pivot columns of B (RREF of A) $= \text{rank}(A)$.

Theorem:

$$\boxed{\dim C(A) = \text{rank}(A)}$$

② Make a basis out of new and "simpler" vectors:

- Arrange $v_1, \dots, v_m$ as rows of a matrix C.
- Reduce C to RREF, called D.
- The nonzero rows of D form a basis of V.

Ex:

$$C = \begin{bmatrix} -1 & -2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = D$$

A basis of V is $\{(1,0,3), (0,1,-2)\}$.

Why does this method work?

$V = \text{span}\{\text{rows of } C\} =$ row space of C, denoted by $R(C)$.

After each row operation, the row space doesn't change.

Observation:

$\dim V = \#$ nonzero rows of D (RREF) of C $= \text{rank } C$.

Theorem:

$$\boxed{\text{For any matrix } A, \quad \dim C(A) = \dim R(A) = \text{rank}(A)}$$

* Compare two methods:

Method 1 provides a basis made out of the original vectors.

Method 2 gives a means how to "complement" a basis of V to get a basis of $\mathbb{R}^n$. In the example, we see that not all columns of D is pivot. To replace the zero row of D by another row to make it row equivalent to I_3, we can choose

$$v = (0,0,1).$$

Then a basis for $\mathbb{R}^3$ is $\{(1,0,3), (0,1,-2), (0,0,1)\}$

Null space of a matrix A is the space $\{x : Ax = 0\}$.

For example,

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -1 & -2 & 1 & 0 \\ -1 & -1 & 4 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Continue next time.