Lecture 15 (10/24/2018)

Linear system of eqs.

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_1 + 3x_2 = 3 \\ x_1 + x_2 + 2x_3 = 1 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & 3 & 0 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

nonpivot column

$x_3 = t$ (free variable)

Second row: $x_2 - x_3 = 1 \Rightarrow x_2 = 1+t$

First row: $x_1 + 3x_3 = 0 \Rightarrow x_1 = -3t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3t \\ 1+t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$$

Null space of a matrix A is defined as $\{x : Ax = 0\}$.
This is indeed a subspace. To determine the null space is equivalent to solving a homogeneous system.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ -2 & -4 & 1 & -4 & 0 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

Null space of A is $\{x \in \mathbb{R}^5 : Ax = 0\}$ (subspace of $\mathbb{R}^5$), denoted by $N(A)$

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 = s$, $x_4 = t$, $x_5 = u$
(free variables)

nonpivot cols.

From the second eq., $x_3 = -2x_4 + 2x_5 = -2t + 2u$.
From the first eq., $x_1 = -2x_2 - 3x_4 + x_5 = -2s - 3t + u$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-3t+u \\ s \\ -2t+2u \\ t \\ u \end{bmatrix} = \underbrace{s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$$

This subspace is spanned by the vectors
$v_1 = (-2, 1, 0, 0, 0)$
$v_2 = (-3, 0, -2, 0, 0)$
$v_3 = (1, 0, 2, 0, 1)$

These vectors are linearly independent. As a rule, the vectors obtained by this procedure are always linearly ind. (you don't need to check their linear ind. in homework or exams), thus form a basis for $N(A)$.
The dimension of $N(A)$ is called nullity of A.

Observation:

$$\begin{aligned} \text{rank}(A) &= \#\text{ pivot cols in RREF of } A = \dim C(A) \\ &= \#\text{ nonzero rows in RREF of } A = \dim R(A) \\ \text{null}(A) &= \dim N(A) = \#\text{ nonpivot cols. of } A \end{aligned}$$

Thus,

$$\boxed{\text{rank}(A) + \text{null}(A) = \#\text{ col. of } A}$$

This is known as the rank-nullity theorem.