Lecture 16 (10/29/2018)

Review:

$$A = \begin{bmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_n \\ | & | & & | \end{bmatrix} \xrightarrow{\text{RREF}} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\text{pivot columns indicate basis of } \underbrace{\operatorname{span}\{C_1, \dots, C_n\}}_{C(A)}}$$

the column space of A

$$\underbrace{\#\text{ pivot cols.}}_{=\operatorname{rank}(A)} = \dim C(A)$$

$$A = \begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ & \vdots & \\ - & R_m & - \end{bmatrix} \xrightarrow{\text{RREF}} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\text{non-zero rows form basis of } \underbrace{\operatorname{span}\{R_1, \dots, R_m\}}_{R(A)}}$$

(the row space of A)

$$\underbrace{\#\text{ nonzero rows}}_{\operatorname{rank}(A)} = \dim R(A)$$

Thus,

$$\boxed{\dim C(A) = \operatorname{rank}(A) = \dim R(A)}$$

Null space of A: $\quad N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$

$$A = \begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ & \vdots & \\ - & R_m & - \end{bmatrix} \xrightarrow{\text{RREF}} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\text{nonpivot cols indicate free var.}}$$

$$\#\text{ nonpivot cols} = \#\text{ free var.} = \dim N(A)$$

Thus,

$$\boxed{\dim N(A) + \operatorname{rank}(A) = \#\text{ cols. of } A}$$ rank-nullity theorem.

Linear maps / transformations:

$f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ (in general $f: \underbrace{V}_{\text{subspace of } \mathbb{R}^m} \longrightarrow \underbrace{W}_{\text{subspace of } \mathbb{R}^m}$)

f is called linear if it satisfies two following conditions:

- additive: $f(x+y) = f(x) + f(y)$
- scalar-multiplicative: $f(cx) = cf(x)$

scalar (number)

Recall: $x = (x_1, \ldots, x_m)$

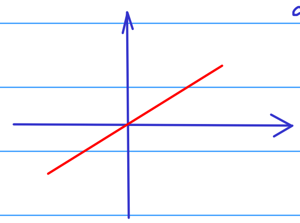
$y = (y_1, \ldots, y_m)$

$x+y = (x_1+y_1, \ldots, x_m+y_m)$

$cx = (cx_1, cx_2, \ldots, cx_m)$

Ex:

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x\underbrace{f(1)}_{a} = ax$



$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\quad f(x_1, x_2) = ?$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2} = x_1 e_1 + x_2 e_2$$

$$f(\underbrace{x_1, x_2}_{x}) = f(x_1 e_1 + x_2 e_2) = x_1 \underbrace{f(e_1)}_{a} + x_2 \underbrace{f(e_2)}_{b}$$

$$f(x_1, x_2) = ax_1 + bx_2.$$

In general, a linear map $f: \mathbb{R}^m \to \mathbb{R}$ is of the form

$$f(x_1, \ldots, x_m) = \underbrace{a_1 x_1 + a_2 x_2 + \cdots + a_m x_m}_{\text{linear comb. of the coordinates}}$$

In matrix form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad f(x) = \underbrace{[a_1 \; a_2 \cdots a_m]}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = Ax$$

In general, if $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map then

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

each is a linear map from $\mathbb{R}^m$ to $\mathbb{R}$.

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \underset{m \times n}{\underset{\smile}{A}} x \qquad A : \text{matrix representing } f$$

Ex:

$f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (a_1 x + a_2 y + a_3 z, \; b_1 x + b_2 y + b_3 z)$

$$f(X) = \begin{bmatrix} a_1 x + a_2 y + a_3 z \\ b_1 x + b_2 y + b_3 z \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = AX$$

Kernel of a linear map:

$f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (2x_1 + 3x_2 - x_3, \; x_1 - x_2)$

$$\ker(f) = \{ x \in \mathbb{R}^3 : f(x) = 0 \} = f^{-1}(\{0\}) \quad (\text{pre-image of } 0)$$

$$= \{ x \in \mathbb{R}^3 : Ax = 0 \}$$

$=$ null space of A .... subspace of $\mathbb{R}^3$

Back to the example: what is a basis and the dimesion of $\ker(f)$?

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -1/5 \end{bmatrix}$$

Continue to find basis of the null space of A ... dimension $= 1$.