# Lecture 17 (10/31/2018)

$f: \mathbb{R}^n \to \mathbb{R}^m$ linear ..... $A$ $m \times n$

$$f(x) = Ax$$

$\uparrow$ matrix multiplication

How to find $A$ from $f$?

$$A = \begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix}$$

Here $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$:

$$e_i = (0, \ldots, \underset{\substack{\uparrow \\ i}}{1}, \ldots, 0)$$

Ex: $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (y, x+z)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Algebra of linear maps:

$f, g: \mathbb{R}^n \to \mathbb{R}^m$ linear

$f \dashrightarrow A$

$g \dashrightarrow B$

$f+g$ is defined as a map that does the following:

$$(f+g)(x) = f(x) + g(x)$$

$cf$ is defined as a map:

$$(cf)(x) = cf(x)$$

Observation:

$f+g \dashrightarrow A+B$

$cf \dashrightarrow cA$

Composition of two linear maps:

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

$g \circ f$ is defined as a map such that $g \circ f(x) = g(f(x))$

$g \circ f$ is also a linear map!

$f \dashrightarrow A$

$g \dashrightarrow B$

$g\circ f \dashrightarrow BA$ (matrix multiplication)

This is why matrix multiplication is defined the way we have learned.

Ex:

$$g: \mathbb{R}^2 \to \mathbb{R}^4, \quad g(x,y) = (x+y,\ x-y,\ y,\ x)$$

$$B = \begin{bmatrix} | & | \\ g(e_1) & g(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R}^4 \end{tikzcd}$$

$$g\circ f \dashrightarrow BA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Kernel of a linear map:

$f: \mathbb{R}^3 \to \mathbb{R}^2$

$g: \mathbb{R}^{100} \to \mathbb{R}^2$

$\Big\}$ loss of information: one can't recover the input from the output.

How to formulate the idea of "loss of info." in the output?

$f(x) = f(y)$ for some $x \neq y$ (when loss of info. occurs)

$\rightsquigarrow f(x-y) = 0$

Put $z = x-y \neq 0$. Then $f(z) = 0$.

The set

$$\{z \in \mathbb{R}^n : f(z) = 0\}$$

seems to be a good set that captures the loss of info. It's called the **kernel** of the linear map $f$.

$$\ker(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$$
$$= \{x \in \mathbb{R}^n : Ax = 0\}$$
$$= N(A) \quad \text{(null space of } A\text{)}$$

$\dim \ker(f) = \dim N(A) = \text{nullity of } A.$

Next time: range of a linear map.