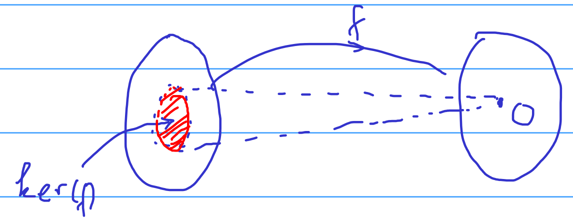
Lecture 18 (11/2/2018)

$f: \mathbb{R}^n \to \mathbb{R}^m$ linear

represented by matrix A of size $m \times n$.

$\ker(f) = N(A)$ (the null space of A)



$\ker(f) = \underbrace{f^{-1}}_{\text{the preimage of } \{0\}}(\{0\})$

f is one-to-one (injective) if and only if $\ker(f) = \{0\}$

If $\ker(f) \neq \{0\}$, f yields "repeated" outputs.

What is $C(A)$ (the column space of A) in terms of f?

Recall:

$$A = \begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix}$$

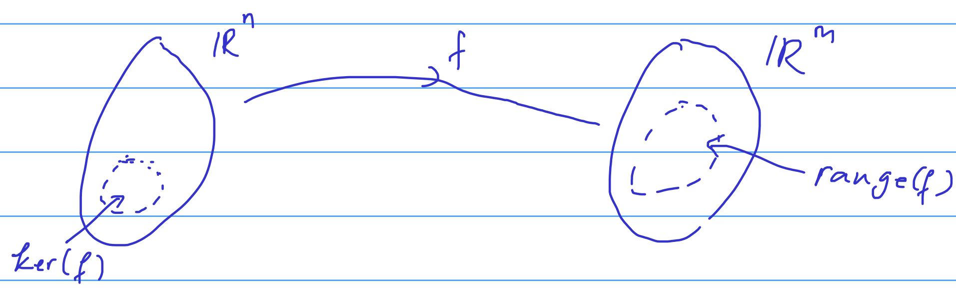
$C(A)$ = set of all linear combinations of $f(e_1), \dots, f(e_n)$.

= set of all vectors of form $c_1 f(e_1) + c_2 f(e_2) + \cdots + c_n f(e_n)$

= " $f(c_1 e_1 + \cdots + c_n e_n)$

= " $f(c_1, c_2, \dots, c_n)$

= range(f)



If range$(f) = \mathbb{R}^m$, f is said to be onto (or surjective).

How to check if a linear map is surjective?

- Write the matrix A representing f.
- Find RREF of A:

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \quad & \quad \end{bmatrix} = B$$

If B has m pivot cols, then $\dim C(A) = m$ (i.e. $C(A) = \mathbb{R}^m$).
In this case, f is surjective.

If B has less than m pivot cols, $C(A)$ is strictly smaller than $\mathbb{R}^m$. Thus, f is not surjective.

Observation:

$$\dim \ker(f) = \dim N(A) = \text{nullity of } A$$

$$\dim \text{range}(f) = \dim C(A) = \text{rank}(A)$$

By rank-nullity theorem:

$$\dim \ker(f) + \dim \text{Range}(f) = \text{nullity}(A) + \text{rank}(A) = \underbrace{n}_{\substack{\text{dimension of the} \\ \text{domain of } f.}}$$

$f: \mathbb{R}^{100} \to \mathbb{R}^{150}$ can't be surjective!

$$A = \underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{100 \text{ cols}} \Big\} \, 150 \text{ rows}$$

The RREF can't have 150 pivot cols.

$f: \mathbb{R}^{150} \to \mathbb{R}^{100}$ can't be injective

Suppose by contradiction that f is injective. Then $\ker(f) = 0$.

$$\underbrace{\dim \ker(f)}_{=0} + \dim \text{Range}(f) = 150$$

Thus, $\text{Range}(f)$ must have dimension 150.

On the other hand, $\text{range}(f)$ is a subspace of $\mathbb{R}^{100}$. Its dimension can't exceed 100. Contradiction!