Lecture 19 (11/05/2018)

f is injective $\iff \ker(f) = \{0\}$

$\iff N(A) = \{0\}$

$\iff$ system $Ax = 0$ has only trivial sol.

How to check if $f: \mathbb{R}^n \to \mathbb{R}^m$ is injective?

- if $n > m$: f is not injective
- if $n \leq m$: $A \xrightarrow{RREF} \begin{bmatrix} \; \end{bmatrix} = B$

If all cols. are pivot: injective

Otherwise: not injective

f is surjective $\iff \text{range}(f) = \mathbb{R}^m$

$\iff C(A) = \mathbb{R}^m$

$\iff$ the system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$.

How to check if $f: \mathbb{R}^n \to \mathbb{R}^m$ is surjective?

- if $n < m$: f is not surjective
- if $n \geq m$: $A \xrightarrow{RREF} \begin{bmatrix} \; \end{bmatrix} = B$

If there are m pivot cols, f is surjective

Otherwise: not surjective

Ex: $f: \mathbb{R}^3 \to \mathbb{R}^2 \quad f(x,y,z) = (x+2y+3z,\ 2x+4y+6z)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Find kernel:

$y = t,\ z = s,\quad x = -2t - 3s$

$$\ker f = \left\{ \begin{bmatrix} -2t-3s \\ t \\ s \end{bmatrix} : t,s \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t,s \in \mathbb{R} \right\}$$

This is a 2-dim space with basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find range :

$C(A)$ is the space spanned by the first column of A :

$$\text{Range}(f) = C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$$

This is a 1-dimensional space with basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

* The case $n = m$: $\quad f : \mathbb{R}^n \to \mathbb{R}^n$

know : $\dim \ker(f) + \dim \text{range}(f) = n$ (rank-nullity theorem)

$$\underbrace{\ker(f) = \{0\}}_{f \text{ injective}} \iff \underbrace{\text{range}(f) = \mathbb{R}^n}_{f \text{ surjective}}$$

injective $\iff$ surjective $\iff$ bijective

How to check if f is inj / sur / bij ?

$$A = \begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n$$

f is inj / sur / bij $\iff$ A is invertible $\iff$ $\det(A) \neq 0$.

Ex :

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (x + y,\ y + z,\ z + x)$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \det(A) = 2 \neq 0$$

f is bijective.

$f^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is a map such that $f \circ f^{-1} = f^{-1} \circ f = Id_{\mathbb{R}^3}$.

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$f^{-1}(x, y, z) = \left(\tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}z,\ \tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}z,\ -\tfrac{1}{2}x + \tfrac{1}{2}y + \tfrac{1}{2}z \right).$$

A picture of $N(A)$, $C(A)$, $R(A)$: (side notes, not on exam)

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear ..... A $m \times n$

$$A = \begin{bmatrix} \text{---}\ w_1\ \text{---} \\ \text{---}\ w_2\ \text{---} \\ \vdots \\ \text{---}\ w_m\ \text{---} \end{bmatrix}$$

Consider $v \in N(A)$ (the kernel of f)

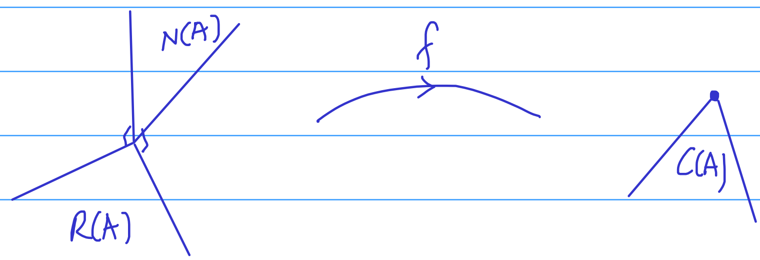
$$Av = \begin{bmatrix} \text{---}\ w_1\ \text{---} \\ \text{---}\ w_2\ \text{---} \\ \vdots \\ \text{---}\ w_m\ \text{---} \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} = \begin{bmatrix} w_1 \cdot v \\ w_2 \cdot v \\ \vdots \\ w_m \cdot v \end{bmatrix}$$

We know that this must be equal to zero.

$w_1, w_2, \ldots, w_m$ are perpendicular to v.

$R(A) \perp N(A)$: the row space is the orthogonal complement of the null space in $\mathbb{R}^n$.

$$\dim R(A) + \dim N(A) = n$$



$$N(A) \xrightarrow{f} \{0\}$$

$$R(A) \xrightarrow{f} C(A)$$

The restriction of f on $R(A)$ is bijective.

Coordinates of a vector:

$x = (2, 1, 3)$ ←---- these numbers are coordinates of x (as a geometric point/vector in 3D-space)

$x = 2e_1 + e_2 + 3e_3$

$S_0 = \{e_1, e_2, e_3\}$ - standard basis of $\mathbb{R}^3$

$$[x]_{S_0} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Consider:

$$\left.\begin{array}{l} v_1 = (1, 0, 1) \\ v_2 = (0, 1, 1) \\ v_3 = (1, 1, 1) \end{array}\right\} \quad S = \{v_1, v_2, v_3\} \text{ also form a basis for } \mathbb{R}^3.$$

$$[x]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \cdots\cdots \text{ coordinates of } x \text{ in basis } S.$$

means: $x = c_1 v_1 + c_2 v_2 + c_3 v_3$

How to determine c_1, c_2, c_3?

$$\underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}}_{P} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = P^{-1} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

In other words, $\boxed{[x]_S = P^{-1} [x]_{S_0}}$