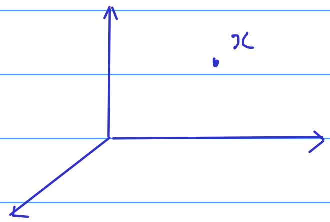
Lecture 20 (11/07/2018)

* Coordinates of a vector:



A <u>point</u> $x \in \mathbb{R}^3$ is represented by 3 numbers called coordinates.

$$\underbrace{x}_{\text{point}} = \underbrace{(x_1, x_2, x_3)}_{\text{label of a point}}$$

What this means is $\quad x = x_1 e_1 + x_2 e_2 + x_3 e_3$

where $S_0 = \{e_1, e_2, e_3\}$ is standard basis of $\mathbb{R}^3$:

$$e_1 = (1, 0, 0)$$
$$e_2 = (0, 1, 0)$$
$$e_3 = (0, 0, 1)$$

Consider another basis of $\mathbb{R}^3$: $\quad S = \{v_1, v_2, v_3\}$

$$v_1 = (1, 0, 1)$$
$$v_2 = (0, 1, 1)$$
$$v_3 = (1, 1, 1)$$

(Why is it a basis?)

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3$$

The triple (c_1, c_2, c_3) is unique, and is called the coordinates of x with respect to basis S.

$$[x]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

How to find c_1, c_2, c_3?

$$\underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[x]_S} = \underbrace{\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{[x]_{S_0}}$$

$$[x]_S = P^{-1} [x]_{S_0}$$

For example, take $x = (3, 2, 1)$:

$$P^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$[x]_S = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

* Matrix representing a map in a basis

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $\quad f(x,y,z) = (x+2y+z,\ x-z,\ y+2z)$

The matrix representing f in basis S, denoted by $[f]_S$ is a matrix such that

$$\boxed{[f(x)]_S = [f]_S\,[x]_S}$$

How to find $[f]_S$? The matrix representing f in standard basis is $[f]_{S_0}$.

$$[f(x)]_{S_0} = [f]_{S_0}\,[x]_{S_0}$$

$$P^{-1}[f(x)]_{S_0} = [f(x)]_S = [f]_S\,[x]_S = [f]_S\,P^{-1}[x]_{S_0}$$

$$\Rightarrow \quad P^{-1}[f]_{S_0}[x]_{S_0} = [f]_S\,P^{-1}[x]_{S_0}$$

$$P^{-1}[f]_{S_0} = [f]_S\,P^{-1}$$

$$\Rightarrow \quad \boxed{[f]_S = P^{-1}[f]_{S_0}\,P}$$

In the previous example:

$$[f]_{S_0} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

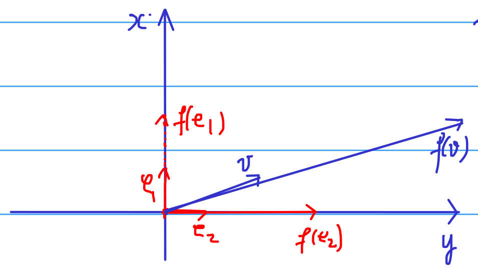
$$[f]_S = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

* The need of change of basis:

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f(x, y) = (2x, 3y)$



f only stretches along the e_1 or e_2 direction: the output and input are parallel to each other.

f doesn't preserve any other directions.

$$\text{ratio} = \frac{\text{output}}{\text{input}}$$

is maximized in e_2-direction, minimized in e_1-direction.

$v = (1, 2)$

(continue next time)