Lecture 21 (11/9/2018)

Review:

$f: \mathbb{R}^n \to \mathbb{R}^n$ linear ..... $A = [f]_{S_0}$ $(n \times n)$

$$A = \begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix} \qquad f(v) = Av$$

$S_0 = \{e_1, e_2, \dots, e_n\}$ - standard basis of $\mathbb{R}^n$.

$S = \{v_1, v_2, \dots, v_n\}$ - another basis of $\mathbb{R}^n$.

$$[f]_S = \begin{bmatrix} | & | & & | \\ [f(v_1)]_S & [f(v_2)]_S & \cdots & [f(v_n)]_S \\ | & | & & | \end{bmatrix} \qquad [f(v)]_S = [f]_S [v]_S$$

$[f]_S = P^{-1} A P$ where $P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

Ex: $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x-y,\ y+2z,\ -x+2y+z)$

$$S = \left\{ \begin{array}{l} v_1 = (1,6,13), \\ v_2 = (1,5,10), \\ v_3 = (1,4,8) \end{array} \right\}$$

what is $[f]_S$?

- $P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 8 \end{bmatrix}$

- $P^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ -4 & 5 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

- $A = [f]_{S_0} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

- $[f]_S = P^{-1} A P = \begin{bmatrix} -40 & -31 & -25 \\ 132 & 103 & 82 \\ -97 & -76 & -60 \end{bmatrix}$

The first column tells us that $f(v_1) = -40v_1 + 132v_2 - 57v_3$

" second " $f(v_2) = -31v_1 + 103v_2 - 76v_3$

.....

* Eigenvalues and eigenvectors:

$f: \mathbb{R}^n \to \mathbb{R}^n$ ..... A $n \times n$

A vector $v \neq 0$ is said to be an eigenvector of f (or A) if $f(v)$ is parallel to v.

In other words, there exists a number λ (could be 0) such that

$$f(v) = \lambda v \quad (\text{or } Av = \lambda v)$$

λ is called eigenvalue, and v is an eigenvector corresponding to λ.

We only talk about eigenvectors/eigenvalues of a square matrix.

How to compute them?

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

$\det(A - \lambda I) = 0 \leftarrow$ Solve for eigenvalues from this equation.

Once λ is found, solve for v: the set of all eigenvectors is the null space of $A - \lambda I$ (the origin excluded). The set of all eigenvectors corresponding to λ, including the zero vector, is called the eigenspace corresponding to λ, denoted by $E(\lambda)$.

Ex:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

$$A - \lambda I_2 = \begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (3-\lambda)^2 - 2^2 = (1-\lambda)(5-\lambda)$$

Two roots are $\lambda_1 = 1$, $\lambda_2 = 5$.

* Find the eigenvectors corresponding to λ_1:

$$A - I_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Null space: $E(\lambda_1) = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

This is the set of all eigenvectors of λ_1 (including 0-vector).
It is a 1-dimensional subspace of $\mathbb{R}^2$, with basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

* Find the eigenvectors corresponding to λ_2:

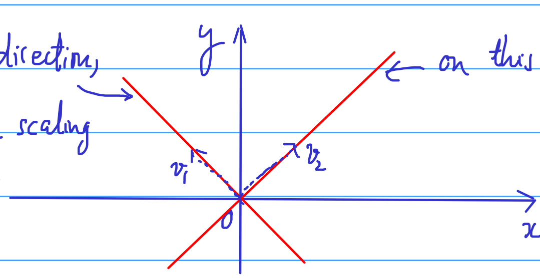
$$A - 5I_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Null space: $E(\lambda_2) = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

It is a 1-dimensional subspace of $\mathbb{R}^2$, with basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Picture:

On this direction, f acts as a scaling by factor 1.



on this direction, f acts as a scaling by factor 5.