Lecture 22 (11/14/2018)

Review:

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \text{linear}$$

S --- basis of $\mathbb{R}^n$

$$\begin{tikzcd}
x \arrow[r, "f"] & y \arrow[d, dotted, "P^{-1}"] \\
{[x]_S} \arrow[u, dotted, "P"] \arrow[r, "{[f]_S}"] & {[y]_S}
\end{tikzcd}$$

f in standard basis S_0

$$y = [f]_{S_0} x$$

f in basis S

$$[y]_S = [f]_S [x]_S$$

$$x = P[x]_S$$

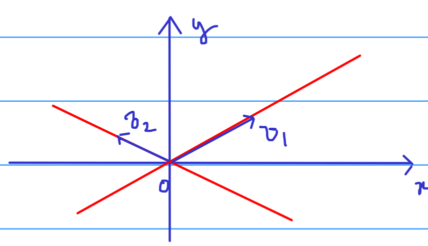
$$y = P[y]_S$$

$$\Rightarrow \quad [f]_S = P^{-1} [f]_{S_0} P$$

From the identity $[f(x)]_S = [f]_S [x]_S$, we deduce that

$$[f]_S = \begin{bmatrix} | & & | \\ [f(v_1)]_S & \cdots & [f(v_n)]_S \\ | & & | \end{bmatrix}$$

Ex: $f : \mathbb{R}^2 \to \mathbb{R}^2$

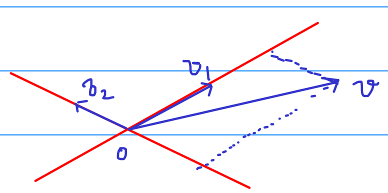


$v_1 = (3, 2)$

$v_2 = (-1, 1)$

Suppose $f(v_1) = 5v_1$ and $f(v_2) = -3v_2$.

In other words, f acts by scaling by factor 2 along v_1-direction, and by factor -1 along v_2-direction.



For $v = 2v_1 - v_2$,

$$f(v) = 5.(2v_1) - (-1)v_2 = 10v_1 + v_2$$

Put $S = \{v_1, v_2\}$. Then

$$[f]_S = \begin{bmatrix} | & | \\ [f(v_1)]_S & [f(v_2)]_S \\ | & | \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$$

How to find f in standard basis?

$$A = [f]_{S_0}$$

We have $\quad [f]_S = P^{-1} A P$, with $P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

Then $\quad A = P [f]_S P^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 9 & 24 \\ 16 & 1 \end{bmatrix}$

$$f(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9/5\, x + 24/5\, y \\ 16/5\, x + 1/5\, y \end{bmatrix}$$

* Eigenvalues & eigenvectors:

Matrix A ... $n \times n$

$$E(\lambda) = \{ v \in \mathbb{R}^n : Av = \lambda v \}$$

Ex:

$A = \begin{bmatrix} 21 & -24 \\ 16 & -19 \end{bmatrix}$ .... matrix representation of some linear map $f: \mathbb{R}^2 \to \mathbb{R}^2$

Eigenvalues: $\lambda_1 = 5, \quad \lambda_2 = -3$

Eigenvalues: $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E(\lambda_1) = \operatorname{span}\{v_1\}$

$E(\lambda_2) = \operatorname{span}\{v_2\}$

→ two lines on the plane

$$\underbrace{\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}}_{\text{diagonal matrix}} = [f]_S = P^{-1} A P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 21 & -24 \\ 16 & -19 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

algebraic picture

To diagonalize a matrix A is to find matrices P and D such that:

- D is diagonal,
- $D = P^{-1} A P$

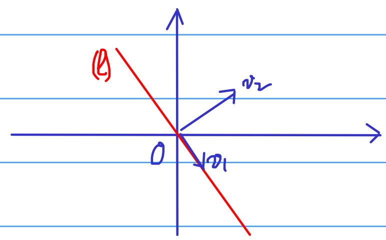
geometric picture

In "linear map" picture: to diagonalize a linear map $f: \mathbb{R}^n \to \mathbb{R}^n$ is to find a basis $S = \{v_1, v_2, \dots, v_n\}$ such that f acts as a scaling on each direction $v_1, v_2, \dots, v_n$.

Each direction $v_1, v_2, \dots, v_n$ is called an eigenvector. The scaling factor that f acts on each direction is called eigenvalue.

Ex: Consider the orthogonal projection onto the line

$(\ell) : 3x + 2y = 0$



$f: \mathbb{R}^2 \to \mathbb{R}^2$ maps a vector to its projection on (ℓ).

$v_1 = (2, -3)$ lies on (ℓ)

$f(v_1) = v_1$ ⟵ this is an eigenvector

(a direction along which f acts by scaling)

$v_2 = (3, 2)$ is perpendicular to (ℓ)

$f(v_2) = 0 = 0 . v_2$

v_2 is another eigenvector.

$$\underbrace{f(v_1) = v_1}_{\text{eigenvalue } \lambda_1 = 1}, \qquad \underbrace{f(v_2) = 0}_{\text{eigenvalue } \lambda_2 = 0}$$

Thus, $\mathcal{B} = \{v_1, v_2\}$ is a basis of $\mathbb{R}^2$ that diagonalizes f.

$$[f]_{\mathcal{B}} = \begin{bmatrix} | & | \\ [f(v_1)]_S & [f(v_2)]_S \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is f in standard basis?

$$\begin{tikzcd} x \arrow[r, "A"] \arrow[d, dashed, "P^{-1}"'] & y \\ {[x]_S} \arrow[r, "{[f]_S}"] & {[y]_S} \arrow[u, dashed, "P"'] \end{tikzcd}$$

f in S_0

f in S

According to the diagram, $A = P\,[f]_S\,P^{-1} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}^{-1}$

$$= \frac{1}{13}\begin{bmatrix} 4 & -6 \\ -6 & 9 \end{bmatrix}$$

Thus, $f(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{13}x - \frac{6}{13}y \\ \frac{-6}{13}x + \frac{9}{13}y \end{bmatrix}$.