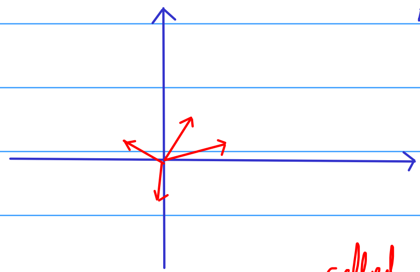
Lecture 23 (11/16/2018)

Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$ identity map

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Every direction is preserved under f. All vectors are eigenvectors.

Check with computation:

* Find the eigenvalues of A:

called characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

This polynomial has double root $\lambda_1 = \lambda_2 = 1$.

* Find eigenvectors of λ_1:

$$A - \lambda_1 I = A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The null space $= \mathbb{R}^2$.

$$E(\lambda_1) = \mathbb{R}^2 = \operatorname{span}\{e_1, e_2\}$$

The matrix (or the linear map) is diagonalizable.

Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \left(\sim f: \mathbb{R}^2 \to \mathbb{R}^2,\ f(x,y) = (x+2y,\ y)\right)$$

Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$ ... double root

$\lambda_1 = \lambda_2 = 1$

General rule:

If A is a upper (or lower) triangular matrix then the eigenvalues of A are the entries on the diagonal.

Eigenvectors: $A - \lambda I = A - I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

nonpivot column

$x_1 = t,\ x_2 = 0$

$$E(\lambda) = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

↑ the set of eigenvectors corresponding to λ

We need 2 linearly independent vectors to form a basis of $\mathbb{R}^2$. Here we have only one. Thus, there is no basis of $\mathbb{R}^2$ that contains only eigenvectors. A (or f) is not diagonalizable.

To diagonalize a matrix A is to find matrices P and D such that

- P is invertible, D is diagonal
- $D = P^{-1} A P$.

In terms of linear maps, to diagonalize a linear map is to find a basis $S = \{v_1, v_2, \ldots, v_n\}$ such that f acts on each direction $v_1, v_2, \ldots, v_n$ as scalings. The eigenvalues are the scaling factors.

The connection between two pictures: P is the matrix whose columns are vectors in S. And D is a diagonal matrix whose entries on the diagonal are scaling factors.

Suppose we have a polynomial $Q(\lambda) = (1-\lambda)^4 (2-\lambda)^3$. This polynomial has only two distinct roots: $\lambda = 1$ with multiplicity 4,

$\lambda = 2$ with multiplicity 3.

General rule:

The number of linearly ind. eigenvectors corresponding to an eigenvalue does not exceed the multiplicity of the eigenvalue in the characteristic polynomial.

For example, suppose the matrix Q above is the characteristic of a 7×7 matrix (or a linear map $f: \mathbb{R}^7 \to \mathbb{R}^7$). Then $E(1)$ has dimension ≤ 4, and $E(2)$ has dimension ≤ 3.

Theorem: If the dimension of the eigenspace $E(\lambda)$ is equal to the multiplicity of λ for each eigenvalue λ, then the given matrix is diagonalizable. Otherwise, it is not diagonalizable.

* Procedure to diagonalize a matrix A (size $n \times n$)

1) Compute the eigenvalues:

If the matrix is already in upper (or lower) triangular form, then the eigenvalues are the entries on the diagonal. For example, if

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then the eigenvalues of A are $\lambda_1 = 1$ (with multiplicity 3) and $\lambda_2 = 2$ (with mult. 1)

If the matrix is not in upper (or lower) triangular form, then, compute the characteristic polynomial

$$Q(\lambda) = \det(A - \lambda I).$$

- Factor this polynomial to find roots (with multiplicity).

2) To each eigenvalue λ, find the corresponding eigenvectors. This is to find the null space of $A - \lambda I$.

Select a basis for $E(\lambda)$.

If the dimension of $E(\lambda)$ is <u>less</u> than the multiplicity of λ then stop: the matrix is <u>not</u> diagonalizable.

3) Put together the basis of $E(\lambda)$ to obtain the matrix $\underline{P}$.

4) Matrix D is determined by the eigenvalues corresponding to the columns of $\underline{P}$.

* We will consider a few examples next time.