Lecture 24 (11/19/2018)

Diagonalize matrix:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Characteristic polynomial: $\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$

Eigenvalues: $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$

Find eigenvectors of λ_1:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

nonpivot col.

$$x_1 = t,\ x_2 = 0,\ x_3 = 0$$

$$E(\lambda_1) = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

v_1

Find eigenvectors of λ_2:

$$A - \lambda_2 I = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

non pivot column

$$x_2 = t,\ x_3 = 0,\ x_1 = 0$$

v_2

$$E_2(\lambda) = \left\{ \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Find eigenvectors of λ_3:

$$A - \lambda_3 I = \begin{bmatrix} -2 & 0 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

nonpivot

$$x_3 = t,\ x_2 = t,\ x_1 = -\frac{3}{2}t$$

$$E(\lambda_3) = \left\{ \begin{bmatrix} -3/2\,t \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \underbrace{\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}}_{v_3} \right\}$$

$*$ Conclusion :

$$D = P^{-1} A P$$

where $$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

Application: what is A^{100} ?

$$A = PDP^{-1} \Rightarrow A^{100} = \underline{P}\, D^{100} \underline{P}^{-1} = P \begin{bmatrix} 1 & & \\ & 2^{100} & \\ & & 3^{100} \end{bmatrix} P^{-1}$$

Ex

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5$$

Set $\det(A - \lambda I) = 0$ to find all eigenvalues:

$$\lambda = 1 \pm \sqrt{-4} = 1 \pm 2i$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 - 2i \\ \lambda_2 &= 1 + 2i \end{aligned}}$$

λ_1 and λ_2 are complex conjugates of each other

How to find the eigenvectors? work with complex numbers.

$$A - \lambda_1 I = \begin{bmatrix} 1+2i & 5 \\ -1 & -1+2i \end{bmatrix}$$

$$\xrightarrow[R_2 = R_2 + R_1]{R_1 = R_1/(1+2i)} \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$$

$$\left(\frac{5}{1+2i} = \frac{5(1-2i)}{(1+2i)(1-2i)} = \frac{5(1-2i)}{1^2+2^2} = 1-2i \right)$$

$y = t$

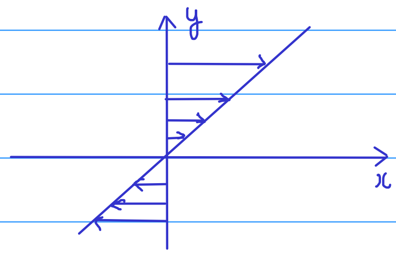
$x = -(1-2i)t$

$$E(\lambda_1) = \left\{ t \begin{bmatrix} -1+2i \\ 1 \end{bmatrix} : t \in \mathbb{C} \right\}$$

$$E(\lambda_2) = \left\{ t \begin{bmatrix} -1-2i \\ 1 \end{bmatrix} : t \in \mathbb{C} \right\}$$

* Connection to linear maps:

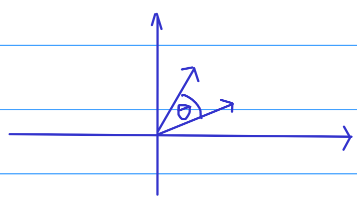
$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x+y, y)$



This linear map has only one eigenvector, which is e_1 (the horizontal direction).

It is not diagonalizable.

The rotation by angle θ is a linear map. It doesn't preserve any directions (unless for $\theta = k\pi$, k integer).



It doesn't have any eigenvectors (that can be visualized). However, it has complex-valued eigenvectors and eigenvalues.

Consider a simple rotation with $\theta = 90°$:

$f(e_1) = e_2$

$f(e_2) = -e_1$

Thus, $$[f]_{S_0} = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

* Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

It has two roots: $\lambda_1 = i$, $\lambda_2 = -i$

* Find eigenvectors corresponding to λ_1:

$$A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow[R_2 = R_2 - R_1]{R_1 = R_1/(-i)} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

nonpivot column

$x_2 = t$

$x_1 = it$

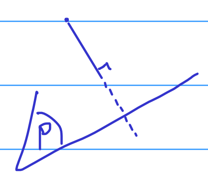
$$E(\lambda_1) = \left\{ \begin{bmatrix} it \\ t \end{bmatrix} : t \in \mathbb{C} \right\} = \operatorname{span}\left\{ \overbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}^{v_1} \right\}$$

* Find eigenvectors corresponding to λ_2:

One can repeat the procedure as above, or say that because $\lambda_2 = \bar{\lambda}_1$ (complex conjugate), $\bar{v}_2 = v_1$.

$$E(\lambda_2) = \left\{ \begin{bmatrix} -is \\ s \end{bmatrix} : s \in \mathbb{C} \right\} = \operatorname{span}\left\{ \overbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix}}^{v_2} \right\}$$

▫ Answer to question regarding to homework:



(P): $x - 2y + 3z = 0$

f : mirror reflection with respect to the plane.

We want to find 3 special directions:

$$\left.\begin{array}{l} v_1 = (-1, 1, 1) \\ v_2 = (2, 1, 0) \end{array}\right\} \text{ lie on } (P)$$

$v_3 = (1, -2, 3)$ --- perpendicular to (P)

$f(v_1) = v_1$

$f(v_2) = v_2$

$f(v_3) = -v_3$

$S = \{v_1, v_2, v_3\}$. Then

$$\left.\begin{array}{l} [f(v_1)]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ [f(v_2)]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ [f(v_3)]_S = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{array}\right\} \Rightarrow [f]_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The matrix representing f in standard basis is

$$A = P[f]_S P^{-1} = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}^{-1}$$

$$\begin{tikzcd} x \arrow[r, "A"] \arrow[d, "P^{-1}"'] & y \\ {[x]_S} \arrow[r, "{[f]_S}"] & {[y]_S} \arrow[u, "P"'] \end{tikzcd}$$