Lecture 25 (11/21/2018)

When do we know if a matrix is diagonalizable?

Recall: for matrix A to be diagonalizable, the dimension of each eigenspace must be equal to the multiplicity of λ.

So if the multiplicity of each λ is 1, the matrix is diagonalizable.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}$$ eigenvalues: $1, 3, -1$ each with multiplicity 1.

Thus, A is diagonalizable.

Ex:

$$B = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$ eigenvalues: $\underset{\text{mult.}=2}{1}, 2$

We can't conclude at this stage whether B is diagonalizable.

One needs to check if $\dim \underbrace{E(1)}_{\text{null space of } B-I} = 2$.

$$B - I = \begin{bmatrix} 0 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

there is only one nonpivot column

$\Rightarrow$ the null space of $B-I$ is only 1-dimensional

$\Rightarrow$ the matrix B is not diagonalizable.

Ex:

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ eigenvalues $= \underset{\text{multi}=2}{1}, 2$

$$C - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

there are two nonpivot colu

$\Rightarrow$ null space $C-I$ has dimension 2.

C is diagonalizable.

Note:

If $A \sim B$ (A is row equivalent to B) then A and B have the same null space and row space. But they aren't necessarily diagonalizable at the same time.

For example,

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{not diagonalizable}} \sim B = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{diagonalizable}}$$

Row operations generally change the eigenvalues/eigenvectors.

Applications:

* Power of matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

What is A^n?

Diagonalize A: $\lambda_1 = -1, \quad \lambda_2 = 5$

$$v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

We have $D = P^{-1}AP$. Thus, $A = PDP^{-1}$

$$A^n = PD^nP^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 5^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2(-1)^n & 5^n \\ -(-1)^n & 5^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2(-1)^n + 5^n & 2(-1)^n + 2.5^n \\ -(-1)^n + 5^n & (-1)^n + 2.5^n \end{bmatrix}$$

* General formula of recursive sequences:

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, ...

$$\begin{cases} x_0 = 1, \ x_1 = 1 \\ x_{n+1} = x_n + x_{n-1} \end{cases}$$

How to get a formula for x_n in terms of n?

Observe:

$$x_{n+1} = 1 \cdot x_n + 1 \cdot x_{n-1}$$

$$x_n = 1 \cdot x_n + 0 x_{n-1}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}}_{X_{n+1}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}}_{X_n}$$

$$\Rightarrow X_{n+1} = A X_n$$

Apply this rule many times: $X_{n+1} = A X_n = A A X_{n-1} = A A A X_{n-2}$

$$= \cdots$$

$$= A^n X_1$$

Thus, $$X_{n+1} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The problem now becomes how to compute the n'th power of A.

This is done by diagonalizing A.