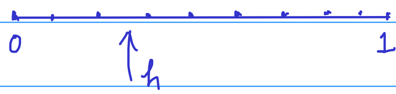
Lecture 26 (11/26/2018)

Application of matrix diagonalization in differential equations.

How do we solve $\dot{x} = x$?



Let $x(0) = 1$. What is $x(1)$?

Discretize the interval $[0,1]$ into segments of length h (step size).

Denote $x_n = x(t_n)$ ..... x at step n

$$x' \approx \frac{x_{n+1} - x_n}{h}$$ ...... x' (approximated) at step n

At step n: $$\frac{x_{n+1} - x_n}{h} = x_n$$

$$\Rightarrow \quad x_{n+1} = (1+h)\, x_n$$

Put $N = 1/h$ .... this is the number of time steps from $t=0$ to $t=1$.

$$x(1) \approx x_N = (1+h)^N \underbrace{x_0}_{\text{the initial cond. } x(0)}$$

$$= (1+h)^{\frac{1}{h}}\, x_0$$

Recall:

$$\lim_{h \to 0^+} (1+h)^{\frac{1}{h}} = e$$

Then $x(1) = e\, x_0$. In general, $x(t) = e^t x_0$.

* Consider system of differential equations:

$$\begin{cases} x_1' = x_1 + 4x_2 \\ x_2' = 2x_1 + 3x_2 \end{cases}$$

This type of systems is quite common in Mathematical Biology.

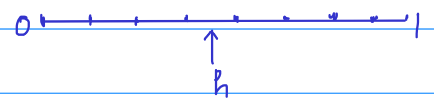
x_1 .... population of species (bacteria) 1, evolve with time.

x_2 "   2   "

The two species interact with each other. The population of one species supports the other.

How to solve?

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow X' = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}}_{A} X$$



What is $X(1)$?

We repeat the above method.

$$\frac{X_{n+1} - X_n}{h} = A X_n$$

$$\Rightarrow X_{n+1} = (I + hA) X_n$$

Then

$$X_N = (I+hA) X_{N-1} = \cdots = (I+hA)^N X_0 = (I+hA)^{\frac{1}{h}} X_0$$

How to take the limit $h \to 0^+$?

Suppose A is diagonalizable. $D = P^{-1} A P$

$$\Rightarrow A = P D P^{-1}$$

$$I + hA = P I P^{-1} + P h D P^{-1} = P(I + hD) P^{-1}$$

$$\Rightarrow (I+hA)^N = P (I + hD)^N P^{-1}$$

$$= P \begin{bmatrix} (1+h\lambda_1)^{1/h} & 0 \\ 0 & (1+h\lambda_2)^{\frac{1}{h}} \end{bmatrix} P^{-1}$$

$$\xrightarrow{h \to 0} P \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} P^{-1}$$

Thus, $X(1) = P \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} P^{-1} X_0$

In general, $\boxed{X(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1} X_0.}$

In this problem, $\lambda_1 = -1$, $\lambda_2 = 5$

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then

$$x(t) = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} x_0$$