Lecture 8 (10/18/2018)

Review.

$A \sim I_n \Rightarrow$ A has an inverse, denoted by A^{-1}: $AA^{-1} = A^{-1}A = I_n$

What if $A \not\sim I_n$?

$A \sim \underset{\text{not } I_n}{\underbrace{B}}$ (RREF of A)

..... B must have a zero row at the bottom.

Consider an example of 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = B$$

Remember how we find the right inverse of A.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightsquigarrow$ two systems of equations. We solve them at once:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ \underbrace{0 \quad 0}_{} & & -2 & 1 \end{array}\right]$$

the first system has no sols. because of row $[0\ 0 \mid -2]$

Similar reason holds for matrix A of size $n\times n$.

* Note: so far we have learned at least 2 methods to solve a system of linear equations $AX = b$:

solve for each unknown, "one at a time" → • Augmented matrix $[A|b]$ — REF (Gauss elimination) / RREF (Gauss-Jordan eli.)

solve for all unknowns "at once". → • (If A is invertible) $X = A^{-1}b$

Determinant:

Consider some motivations of determinants. Suppose we want to find the inverse matrix of:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$[A \mid I_2] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[R_2 = R_2 - cR_1]{R_1 = R_1/a} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right] \to \dots$$

We assume every division is valid. We only want to know what the inverse looks like. The result is

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The number $ad-bc$ plays as an indicator for the invertibility

A. If $ad-bc = 0$ .... not invertible!

$ad-bc \neq 0$ ..... invertible.

We want to introduce a number associated with a matrix to indicate whether the matrix is invertible. Specifically,

If $\det = 0$: not invertible

$\det \neq 0$: invertible

If $A \sim B$, $\det(A)$ and $\det(B)$ should be both zero or nonzero.

$$\left.\begin{array}{l} R_i = cR_i \\ R_i \leftrightarrow R_j \\ R_i = R_i + cR_j \end{array}\right\}$$ determinant should not change from zero to nonzero or vice versa after elementary row operation.

Def: A function that takes in a matrix and produces a number, satisfying 3 following properties:

multi-linear → (1) acts like a linear function on each row:

$$\det\begin{pmatrix} \cdots \\ cR_i \\ \cdots \end{pmatrix} = c\det\begin{pmatrix} \cdots \\ R_i \\ \cdots \end{pmatrix}$$

$$\det\begin{pmatrix} \cdots \\ R_i + R_i' \\ \cdots \end{pmatrix} = \det\begin{pmatrix} \cdots \\ R_i \\ \cdots \end{pmatrix} + \det\begin{pmatrix} \cdots \\ R_i' \\ \cdots \end{pmatrix}$$

anti-symmetric → (2) reverse sign when two rows are exchanged.

$$\det\begin{pmatrix} \cdots \\ R_i \\ \cdots \\ R_j \\ \cdots \end{pmatrix} = -\det\begin{pmatrix} \cdots \\ R_j \\ \cdots \\ R_i \\ \cdots \end{pmatrix}$$

normalization → (3) $\det(I_n) = 1$

Properties:

(4) $A \xrightarrow{R_i = cR_i} B \qquad \det(B) = c\det(A)$

$A \xrightarrow{R_i \leftrightarrow R_j} B \qquad \det(B) = -\det(A)$

$A \xrightarrow{R_i = R_i + cR_j} B \qquad$ Claim: $\det(B) = \det(A)$

(5) $\det\begin{pmatrix} \cdots \\ R_i \\ \cdots \\ R_i \\ \cdots \end{pmatrix} \overset{R_i \leftrightarrow R_j}{=\!=} -\det\begin{pmatrix} \cdots \\ R_i \\ \cdots \\ R_i \end{pmatrix} = 0$

Now go back to (4):

$$\det(B) = \det\begin{pmatrix} \cdots \\ R_i \\ \cdots \\ R_j \\ \cdots \end{pmatrix} + \det\begin{pmatrix} \cdots \\ cR_j \\ \cdots \\ R_j \\ \cdots \end{pmatrix} = \det(A) + c\underbrace{\det\begin{pmatrix} \cdots \\ R_j \\ \cdots \\ R_j \\ \cdots \end{pmatrix}}_{=0} = \det(A)$$

(6) $\det\begin{pmatrix} \boxed{\quad * \quad} \\ 0 \; 0 \; \cdots \; 0 \end{pmatrix} = 0$

why? $A \xrightarrow{R_n = 2R_n} \tilde{A} = A$

$\det(A) = \det(\tilde{A}) = 2\det(A)$.

Thus, $\det(A) = 0$

(7) If $A \not\sim I_n$ (i.e. not invertible) then

$A \sim \underbrace{B}_{\text{RREF of } A} \neq I_n$

B must have a zero row at the bottom.

$\Rightarrow \det(B) = 0$

$B \longrightarrow \cdots \longrightarrow \cdots \longrightarrow A$ (B becomes A through a chain of row operations)

One can trace how the determinant of B is changed after each step

- scaled by a nonzero factor
- reversed its sign
- stay the same

If we start with B with $\det(B) = 0$, we must end up with $\det(A) = 0$.

(8) If $A \sim I_n$ (i.e. A is invertible)

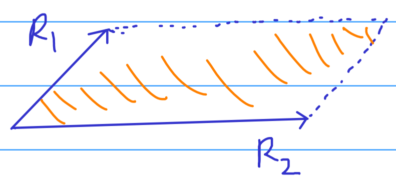
$I_n \longrightarrow \cdots \longrightarrow \cdots \longrightarrow A$: chain of ele. row operations.

If we start with I_n with $\det(I_n) = 1 \neq 0$, we must end up with $\det(A) \neq 0$.

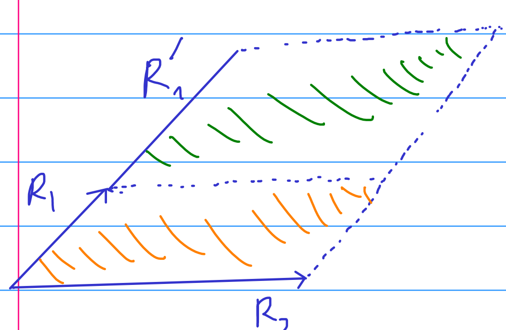
Note that by tracing the determinant at each step, we can compute the determinant of A.

Geometric interpretation

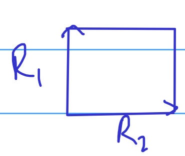
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$



area of the parallelogram $= ad - bc$ (well, strictly speaking $|ad-bc|$)



$$\det\begin{pmatrix} R_1 + R_1' \\ R_2 \end{pmatrix} = \text{area of the large parall.} = \text{sum of two parall.} = \det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \det\begin{pmatrix} R_1' \\ R_2' \end{pmatrix}$$



$$I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

$\det(I_n) =$ area of the square $= 1$