Lecture 9 (10/10/2018)

det: takes in a matrix, produces a number.

$$\det\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

- multilinear ... linear w.r.t each row.
- antisymmetric ---- signed area / volume
- normalization ----- volume of unit cube is 1.

$A \xrightarrow{R_i = cR_i} B \qquad \det(B) = c\det(A)$

$A \xrightarrow{R_i \leftrightarrow R_j} B \qquad \det(B) = -\det(A)$

$A \xrightarrow{R_i = R_i + cR_j} B \qquad \det(B) = \det(A)$

From here one can compute determinant of all matrices.

$$\boxed{A \text{ invertible} \iff \det(A) \neq 0}$$

Examples:

① Elementary matrices:

$R_i = cR_i$ ----- E_1 $\qquad \det(E_1) = c\det(I_n) = c$

$R_i \leftrightarrow R_j$ .... E_2 $\qquad \det(E_2) = -\det(I_n) = -1$

$R_i = R_i + cR_j$ ..... E_3 $\qquad \det(E_3) = \det(I_n) = 1$

$A \xrightarrow{R_i = cR_i} B \qquad B = E_1 A, \ \det(B) = c\det(A)$

$= \det(E_1)\det(A)$

----- $\det(E_i A) = \det(E_i)\det(A)$

Consequence: $\boxed{\det(AB) = \det(A)\det(B)}$

Why? Let's assume A and B are invertible.

optional

$A = E_k \cdots E_2 E_1, \quad B = F_m \cdots F_2 F_1$

$\det(A) = \det(E_k)\det(E_{k-1} \cdots E_2 E_1) = \cdots = \det(E_k)\cdots\det(E_1)$

$\det(B) = \cdots = \det(F_m) \cdots \det(F_1)$

Application:

A invertible .... $AA^{-1} = I_n$

$$\Rightarrow \underbrace{|AA^{-1}|}_{|A|\,|A^{-1}|} = |I_n| = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|}$$

* Geometric interpretation of the product law:

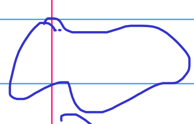
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \cdots \quad f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\quad f \quad} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x,y) = (\underbrace{x+2y}_{u}, \underbrace{3x+4y}_{v})$$

f can be viewed as a change of variable in $\mathbb{R}^2$.

Recall from Calculus:



D

$$\int_{f(D)} g(y)\,dy = \int_D g(f(x)) \underbrace{\det\begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}}_{\text{Jacobian matrix of the transformation}} dx$$

$$u = x + 2y \Rightarrow \nabla u = (1, 2)$$

$$v = 3x + 4y \Rightarrow \nabla v = (3, 4)$$

$$\begin{bmatrix}\nabla u\\ \nabla v\end{bmatrix} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = A$$

Choose $g \equiv 1$ (constant function):

$$\underbrace{\int_{f(D)} 1\, dy}_{\text{area}(f(D))} = \underbrace{\int_{D} |A|\, dx}_{|A|\ \text{area}(D)}$$

$|A|$ --- the scaling of area after the linear transformation.

Now consider functions g and f as follows:

$$\mathbb{R}^2 \underset{B}{\xrightarrow{\ g\ }} \mathbb{R}^2 \underset{A}{\xrightarrow{\ f\ }} \mathbb{R}^2$$

$$D \xrightarrow{\ |B|\ } E \xrightarrow{\ |A|\ } F$$

$$\left.\begin{aligned} \text{area}(E) &= |B|\ \text{area}(D) \\ \text{area}(F) &= |A|\ \text{area}(E) \end{aligned}\right\} \Rightarrow \text{area}(F) = |A|\,|B| \times \text{area}(D).$$

On the other hand,

$$D \underset{AB}{\xrightarrow{\ f\circ g\ }} F \qquad \text{area}(F) = |AB|\ \text{area}(D)$$

$$|AB| = |A|\,|B|$$

Ex $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R_1}{=\!=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$

$$\overset{R_2}{=\!=} \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$\overset{\text{factor}}{=\!=} ac \underbrace{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}_{=0} + ad \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1} + bc \underbrace{\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}}_{=0}$$

$$+ bd \underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{=-1}$$

$$= ad - bc$$