MATH 341, MIDTERM EXAM, FALL 2018

Name	Student ID

- The allowed items for this exam are pencils (or pens), erasers, and calculators.
- You are allowed to use calculator to obtain RREF in Problems 5 and 6. In other problems, you are required to show each elementary row operation to obtain RREF. Nevertheless, you should double check your results with your calculator in all problems.
- The exam has 6 pages (not including this front page). Some problems have more than one part. **Circle your final results.**
- To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little or no credit.

Problem	Possible points	Earned points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

Problem 1. Consider the system of linear equations

$$\begin{cases} x_1 - 6x_2 - 4x_3 &= -5 \\ 2x_1 - 10x_2 - 9x_3 &= -4 \\ -x_1 + 6x_2 + 5x_3 &= 3 \end{cases}$$

(2pt) (a) Write the augmented matrix corresponding to the system.

(8pt) (b) Solve the system by Gauss–Jordan elimination method. You should to show each row operation.

(a)

$$\left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{array}\right] \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[R_2 = \frac{1}{2}R_2]{R_1 = R_1 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 13 \\ 0 & 1 & -1/2 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_1 = R_1 + 7R_3]{R_2 = R_2 + \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

(b)

$x_1 = -1$

$x_2 = 2$

$x_3 = -2$

Problem 2. Consider the system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 & = & -1 \\ -x_1 + 4x_2 + 4x_3 + 3x_4 & = & c \\ -x_1 + 2x_3 + x_4 & = & 2 \end{cases}$$

(5pt) (a) Find all values of c such that the system is consistent (i.e. having at least one solution).

(5pt) (b) For such value(s) of c, solve the system for x_1, x_2, x_3, x_4. If the system has infinitely many solutions, write the solutions in parametric vector form.

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ -1 & 4 & 4 & 3 & c \\ -1 & 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & -1 \\ 0 & 6 & 3 & 3 & c-1 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 = R_3 - \frac{1}{3}R_2 \\ R_2 = R_2/6}]{R_1 = R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -2 \\ 0 & 1 & 1/2 & 1/2 & (c-1)/6 \\ 0 & 0 & 0 & 0 & 1 - \frac{c-1}{3} \end{array}\right]$$

This is the RREF form

The system is consistent if and only if $1 - \frac{c-1}{3} = 0$. This gives $c = 4$.

Now with $c=4$, the RREF becomes

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -2 \\ 0 & 1 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The 3^{rd} and 4^{th} cols. are nonpivot. Thus, $x_3 = s$, and $x_4 = t$ (free variables).

From the second row, $x_2 + \frac{s}{2} + \frac{t}{2} = \frac{1}{2}$ ---> $x_2 = \frac{1}{2} - \frac{s}{2} - \frac{t}{2}$

From the first row, $x_1 - 2s - t = -2$ ---> $x_1 = -2 + 2s + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2+2s+t \\ 1/2 - s/2 - t/2 \\ s \\ t \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}}$$

Problem 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(6pt) (a) Find the inverse of A (if exists) by Gauss–Jordan elimination method. You should show each row operation.

(4pt) (b) Write a chain of elementary operations that takes I_3 to A.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 = R_1 - 2R_2]{R_2 = R_2/3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2/3 & 0 \\ 0 & 1 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1 \\ 0 & 1 & 0 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -2/3 & 1 \\ 0 & 1/3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A \xrightarrow{R_2 = R_2/3} \xrightarrow{R_1 = R_1 - 2R_2} \xrightarrow{R_1 = R_1 + R_3} \xrightarrow{R_2 = R_2 - R_3} I_3$$

$$I_3 \xrightarrow{R_2 = R_2 + R_3} \xrightarrow{R_1 = R_1 - R_3} \xrightarrow{R_1 = R_1 + 2R_2} \xrightarrow{R_2 = 3R_2} A$$

Problem 4. Consider two matrices

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(7pt)(a) Compute the determinant of A and B. You should show how you calculate the determinant. It is not sufficient just to write the result from your calculator.

(3pt)(b) Compute the determinant of matrix $C = 2A^2B^{-3}$.

(a) Because A is an upper triangular matrix, $\det A = 1 \cdot (-1) \cdot (-1) \cdot 1 = 1$

$$|B| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \underline{\underline{R_2 = R_2 - 2R_1}} \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \underline{\underline{R_3 = R_3 + 2R_2}}$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \underline{\underline{R_4 = R_4 - \frac{2}{3}R_3}} \underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1/3 \end{vmatrix}}_{\text{upper triangular matrix}} = 1 \cdot (-1) \cdot 3 \cdot \frac{1}{3} = -1$$

$|A| = 1$, $|B| = -1$

(b) $|C| = |2A^2B^{-3}| = 2^4 |A|^2 |B|^{-3} = 16 \cdot 1^2 \cdot (-1)^{-3} = -16$

$|C| = -16$

Problem 5. In this problem, you are allowed to use calculator to compute RREF. Consider the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}$$

(5pt) (a) Compute the rank of A.

(5pt) (b) Find a basis of the column space of A (i.e. the subspace of $\mathbb{R}^3$ spanned by the columns of A).

(a) $A \xrightarrow[\substack{R_3 = R_3 - 2R_2 \\ R_1 \leftrightarrow R_2}]{R_1 = R_1 + 3R_2} \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_2 = R_2 - 5R_3} \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

There are 2 nonzero rows. Thus, rank$(A) = 2$.

(b) The pivot cols. are 1st and 3rd. A basis for column space of A is

$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ (the 1st and 3rd cols. of A)

Problem 6. In this problem, you are allowed to use calculator to compute RREF. Consider the following vectors

$$\begin{aligned} v_1 &= (1,\ 0,\ 2,\ 1), \\ v_2 &= (-1,\ 1,\ -1,\ -3), \\ v_3 &= (1,\ -2,\ -2,\ 2), \\ v_4 &= (3,\ -4,\ -4,\ 2). \end{aligned}$$

Check if these vectors are linearly independent. If they are linearly dependent, express one vector as a linear combination of the others.

$$\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 2 & -1 & -2 & -4 \\ 1 & -3 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 = R_4 - R_1]{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -4 & -10 \\ 0 & -2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 = R_3 - R_2 \\ R_4 = R_4 + 2R_2}]{R_1 = R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & -3 & -9 \end{bmatrix} \xrightarrow[\substack{R_4 = R_4 + 3R_3 \\ R_1 = R_1 + R_3}]{\substack{R_2 = R_2 - 2R_3 \\ R_3 = R_3 / (-2)}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$ has augmented form

$$\left[\begin{array}{cccc|c} | & | & | & | & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 \\ | & | & | & | & 0 \\ & & & & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

nonpivot column.

v_1, v_2, v_3, v_4 are linearly dependent.

$$\begin{cases} c_4 = t \\ c_3 = -3t \\ c_2 = -2t \\ c_1 = -2t \end{cases}$$

Take $t = 1$: $c_1 = -2$, $c_2 = -2$, $c_3 = -3$, $c_4 = 1$

Thus,

$$-2v_1 - 2v_2 - 3v_3 + v_4 = 0$$

$$v_4 = 2v_1 + 2v_2 + 3v_3.$$