Final Review
Answer Key

1. Let $V = P_2(\mathbb{R})$ be the vector space of all polynnomials of degree ≤ 2 with real coefficients. Let

$$V_1 = \{f \in V : f(1) = 0\},$$
$$V_2 = \{f \in V : f(2) = 0\}.$$

Is $V_1 + V_2$ a direct sum?

Solution: No. One way to prove that $V_1 + V_2$ is not a direct sum of V_1 and V_2 is to show that

$$V_1 \cap V_2 \neq \{0\}.$$

That is, we can try to find a nonzero polynomial $f \in V_1 \cap V_2$. Notice that we want $f(1) = 0$ and $f(2) = 0$, so $(x-1)$ and $(x-2)$ should be factors of $f(x)$. We can set

$$f(x) = (x-1)(x-2) = x^2 - 3x + 2.$$

Then $f \neq 0$ (since, for example, $f(0) = 2$), and $f \in V_1 \cap V_2$. Therefore $V_1 \cap V_2 \neq \{0\}$, so the sum is not direct.

2. Let $V = P_2(\mathbb{R})$. Define $\phi(u) = |u(1)| + |u(2)|$ for any $u \in V$. Is ϕ a norm on V?

Solution: No. To show ϕ is a norm, we need to show three properties:

(i) Positivity: $\phi(u) \geq 0$ for all $u \in V$ and if $\phi(u) = 0$ then $u = 0$.

(ii) Homogeneity: $\phi(\lambda u) = |\lambda|\phi(u)$ for all $u \in V$ and $\lambda \in F$.

(iii) Triangle inequality: $\phi(u + v) \leq \phi(u) + \phi(v)$ for all $u, v \in V$.

In this case, the only thing that is not satisfied is the statement that if $\phi(u) = 0$ then $u = 0$. To see this, let

$$u = (x-1)(x-2) = x^2 - 3x + 2 \in V.$$

Then

$$\phi(u) = |u(1)| + |u(2)| = 0 + 0 = 0,$$

but $u \neq 0$.

3. Let $V = P_2(\mathbb{R})$. Define $\phi(u) = |u(1)| + |u(2)| + |u(3)|$ for any $u \in V$. Show that ϕ is a norm on V.

Solution: To show that ϕ is a norm on V, we need to prove each of the properties listed in the solution to problem 2 above.

(i) Positivity: Let $u \in V$. Then by the definition of absolute value, $|u(1)| \geq 0$, $|u(2)| \geq 0$, and $|u(3)| = 0$, so

$$\phi(u) = |u(1)| + |u(2)| + |u(3)| \geq 0 + 0 = 0.$$

Now suppose $\phi(u) = 0$. Then we must have $u(1) = u(2) = u(3) = 0$. The only way for this to be possible is if $u = 0$ or if $(x-1)$, $(x-2)$, and $(x-3)$ are all factors of u. In the second case, u must be a some multiple of

$$(x-1)(x-2)(x-3)$$

so u would be a polynomial of degree at least 3. Since we assumed $u \in P_2(\mathbb{R})$, this is impossible, so we must have $u = 0$.

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(ii) Homogeneity: Let $\lambda \in F$ and $u \in V$. Then

$$\begin{aligned}\phi(\lambda u) &= |\lambda u(1)| + |\lambda u(2)| + |\lambda u(3)| \\ &= |\lambda||u(1)| + |\lambda||u(2)| + |\lambda||u(3)| \qquad \text{by a property of absolute value} \\ &= |\lambda|\left(|u(1)| + |u(2)| + |u(3)|\right) \\ &= |\lambda|\phi(u),\end{aligned}$$

so homogeneity is satisfied.

(iii) Triangle inequality: Let $u, v \in V$. Then $|u(x) + u(y)| \leq |u(x)| + |u(y)|$ since absolute value satisfies the triangle inequality. Them

$$\begin{aligned}\phi(u+v) &= |u(1) + v(1)| + |u(2) + v(2)| + |u(3) + v(3)| \\ &\leq |u(1)| + |v(1)| + |u(2)| + |v(2)| + |u(3)| + |v(3)| \\ &= (|u(1)| + |u(2)| + |u(3)|) + (|v(1)| + |v(2)| + |v(3)|) \\ &= \phi(u) + \phi(v),\end{aligned}$$

so ϕ satisfies the triangle inequality.

4. Put

$$\begin{aligned}V_1 &= \{A \in M_{2\times 2}(\mathbb{R}) : A = A^T\}, \\ V_2 &= \{A \in M_{2\times 2}(\mathbb{R}) : A = -A^T\}.\end{aligned}$$

Show that $V_1 \oplus V_2 = M_{2\times 2}(\mathbb{R})$.

Solution: Recall the definition of the transpose of a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Now we need to find bases for these two subspaces. First consider V_1:

$$\begin{aligned}V_1 &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right\} \\ &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a = a,\ b = c,\ c = b,\ d = d\right\} \\ &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : b = c\right\} \\ &= \left\{\begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R}\right\} \\ &= \left\{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, d \in \mathbb{R}\right\}\end{aligned}$$

so a basis for V_1 is

$$B_1 = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

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Now consider V_2:

$$\begin{aligned}
V_2 &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a = -a,\ b = -c,\ c = -b,\ d = -d \right\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a = d = 0,\ b = -c \right\} \\
&= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\} \\
&= \left\{ b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : b \in \mathbb{R} \right\}
\end{aligned}$$

so a basis for V_2 is

$$B_2 = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

The set

$$B = B_1 \cup B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

is a spanning set for $V_1 + V_2$. To show that $V_1 + V_2$ is a direct sum we want to prove that B is linearly independent. To do this, write each element of B as a vector in $\mathbb{R}^4$:

$$B = \{(1,0,0,0), (0,1,1,0), (0,0,0,1), (0,1,-1,0)\}.$$

Create a matrix using the elements of B as rows and begin row reduction:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 + R_4 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

It should be obvious to you at this point that this reduces to the 4×4 identity matrix (if it's not obvious, continue reducing). This means that B is linearly independent, so $V_1 + V_2$ is a direct sum.

Since B is a basis for $V_1 \oplus V_2$, we have $\dim(V_1 \oplus V_2) = 4$. Since $V_1 \oplus V_2$ is a subspace of the 4-dimensional space $\mathbb{R}^4$ we must have that $V_1 \oplus V_2 = \mathbb{R}^4$.

5. Let

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 = x_1 + x_3,\ x_3 = 2x_1 - x_2 + 5x_4\}.$$

Find a subspace W of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$.

Solution: First we want to find a basis for V. To do this notice that the conditions of V can be rewritten as

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 - x_2 - x_3 + 5x_4 &= 0. \end{aligned}$$

Written as an augmented matrix, this is

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 5 & 0 \end{array}\right].$$

It is important to note that the rows of this matrix are NOT elements of V.

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Now, perform row reduction:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 5 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 5 & 0 \end{array}\right]$$
$$\xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 5 & 0 \\ 0 & 1 & -3 & 5 & 0 \end{array}\right].$$

This gives the equations

$$\begin{aligned} x_1 &= 2x_3 - 5x_4 \\ x_2 &= 3x_3 - 5x_4, \end{aligned}$$

so

$$\begin{aligned} V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 2x_3 - 5x_4,\ x_2 = 3x_3 - 5x_4\} \\ &= \{(2x_3 - 5x_4, 3x_3 - 5x_4, x_3, x_4) : x_3, x_4 \in \mathbb{R}\} \\ &= \{x_3(2,3,1,0) + x_4(-5,-5,0,1) : x_3, x_4 \in \mathbb{R}\}. \end{aligned}$$

A basis for V is

$$B = \{(2,3,1,0), (-5,-5,0,1)\}.$$

Now form a matrix using the basis vectors as rows:

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ -5 & -5 & 0 & 1 \end{bmatrix}.$$

By adding the rows $(1,0,0,0)$ and $(0,1,0,0)$ we can see that the resulting matrix

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ -5 & -5 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

reduces to the identity matrix. Therefore the set

$$B = \{(2,3,1,0), (-5,-5,0,1), (1,0,0,0), (0,1,0,0)\}$$

is linearly independent. Let

$$\begin{aligned} W &= \operatorname{span}(\{(1,0,0,0), (0,1,0,0)\}) \\ &= \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{R}\}. \end{aligned}$$

Then $\dim(W) = 2$, and B is a basis for $V + W$. Since

$$\dim(V) + \dim(W) = 2 + 2 = 4 = \dim(V + W)$$

this is a direct sum. Now, $V \oplus W$ is a 4-dimensional subspace of the 4-dimensional space $\mathbb{R}^4$, so we must have $V \oplus W = \mathbb{R}^4$.

6. Let V be the vector space of all smooth functions from $\mathbb{R}$ to itself. Let $F\colon V \to V$ be a linear map defined by $F(u) = u' - u$. Let W be the vector space of all smooth functions satisfying the differential equation $u'' + u' + u = 0$. Show that W is invariant under F.

   Solution: Let $w \in W$, so w satisfies

$$w'' + w' + w = 0.$$

   Then $z = F(w) = w' - w$. We want to show that z satisfies $z'' + z' + z = 0$:

$$\begin{aligned} z'' + z' + z &= (w' - w)'' + (w' - w)' + (w' - w) \\ &= w''' - w'' + w'' - w' + w' - w \\ &= (w''' + w'' + w') - (w'' - w' - w) \\ &= (w'' + w' + w)' - (w'' - w' - w) \\ &= (0)' + 0 \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

   Therefore $F(w) = z \in W$, so W is invariant under F.

7. Let $V = M_{2\times 2}(\mathbb{R})$. Let $f\colon V \to V$ be a linear map defined by $f(A) = A^T$. Is f diagonalizable? If it is, find a basis of V in which f is represented by a diagonal matrix.

   Solution: Since f is a map from $M_{2\times 2}(\mathbb{R})$ to $M_{2\times 2}(\mathbb{R})$ we begin by finding a basis for $M_{2\times 2}(\mathbb{R})$ and representing f as a 4×4 matrix. Consider the basis B for $M_{2\times 2}(\mathbb{R})$ given by

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

   Applying f to the elements of B, we get

$$\begin{aligned} f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

   The matrix for f is then

$$A = [f]_{B,B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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We now want to find the eigenvalues of this matrix. Calculate the characteristic polynomial:

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 0-\lambda & 1 & 0 \\ 0 & 1 & 0-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} - 0 + 0 - 0 \\
&= (1-\lambda)\left((-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix} + 0\right) \\
&= (1-\lambda)\left[(-\lambda)(-\lambda)(1-\lambda) - (1-\lambda)\right] \\
&= (1-\lambda)(1-\lambda)(\lambda^2-1) \\
&= (-1)(\lambda-1)(-1)(\lambda-1)(\lambda-1)(\lambda+1) \\
&= (\lambda-1)^3(\lambda+1).
\end{aligned}$$

Setting $(\lambda-1)^3(\lambda+1)$ we get the eigenvalues $\lambda = 1$ and $\lambda = -1$.

Now we want to find the eigenspaces for each eigenvalue. That is, we want to solve

$$(A-\lambda I)v = 0$$

for $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$. First let $\lambda = 1$. The augmented form of $(A-\lambda I)v = 0$ is

$$\left[\begin{array}{cccc|c} 1-1 & 0 & 0 & 0 & 0 \\ 0 & 0-1 & 1 & 0 & 0 \\ 0 & 1 & 0-1 & 0 & 0 \\ 0 & 0 & 0 & 1-1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so $v_2 - v_3 = 0$ and hence $v_2 = v_3$. The eigenvectors v can be written as

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_2 \\ v_4 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace E_1 is

$$B_1 = \{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}.$$

Now let $\lambda = -1$ and follow the same process:

$$\left[\begin{array}{cccc|c} 1+1 & 0 & 0 & 0 & 0 \\ 0 & 0+1 & 1 & 0 & 0 \\ 0 & 1 & 0+1 & 0 & 0 \\ 0 & 0 & 0 & 1+1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so $v_1 = v_4 = 0$ and $v_2 = -v_3$. The eigenvectors v can be written as

$$v = \begin{bmatrix} 0 \\ v_2 \\ -v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

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A basis for the eigenspace E_{-1} is

$$B_{-1} = \{(0, 1, -1, 0)\}.$$

Since

$$\dim(E_1) + \dim(E_{-1}) = 3 + 1 = 4 = \dim(V),$$

the linear map f is diagonalizable. Converting the elements of the set $B_1 \cup B_{-1}$ back into 2×2 form gives

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Then $\mathcal{B}$ is a basis for V under which f is represented by the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$