Some review problems for Final

In the following problems, verify your answer with valid arguments. Make sure to write in full sentences.

1. Let $V = P_2(\mathbb{R})$ be the vector space of all polynomials of degree ≤ 2 with real coefficients. Let

$$\begin{aligned} V_1 &= \{f \in V : \ f(1) = 0\}, \\ V_2 &= \{f \in V : \ f(2) = 0\}. \end{aligned}$$

   Is $V_1 + V_2$ a direct sum ?

2. Let $V = P_2(\mathbb{R})$. Define $\phi(u) = |u(1)| + |u(2)|$ for any $u \in V$. Is ϕ a norm on V ?

3. Let $V = P_2(\mathbb{R})$. Define $\phi(u) = |u(1)| + |u(2)| + |u(3)|$ for any $u \in V$. Show that ϕ is a norm on V.

4. Put

$$\begin{aligned} V_1 &= \left\{A \in M_{2\times 2}(\mathbb{R}) : \ A = A^T\right\} \\ V_2 &= \left\{A \in M_{2\times 2}(\mathbb{R}) : \ A = -A^T\right\}. \end{aligned}$$

   Show that $V_1 \oplus V_2 = M_{2\times 2}$.

5. Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_2 = x_1 + x_3, \ x_3 = 2x_1 - x_2 + 5x_4\}$. Find a subspace W of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$.

6. Let V be the vector space of all smooth functions from $\mathbb{R}$ to itself. Let $F : V \to V$ be a linear map defined by $F(u) = u' - u$. Let W be the vector space of all smooth functions satisfying the differential equation $u'' + u' + u = 0$. Show that W is invariant under F.

7. Let $V = M_{2\times 2}(\mathbb{R})$. Let $f : V \to V$ be a linear map defined by $f(A) = A^T$. Is f diagonalizable? If it is, find a basis of V in which f is represented by a diagonal matrix.