Homework 3

Answer Key

Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a + b + c = 0 \right\},$$

and let W be the set of all functions of the form $\alpha x + \beta x^2 + \gamma e^x$ where $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha + \beta + \gamma = 0$.

1. Show that V is a vector space over $\mathbb{R}$.

   Solution: We will show that V is a subspace of $M_{2\times 2}(\mathbb{R})$. Check the 3 subspace axioms

   (a) **Closed under addition:** Let $A, B \in V$ and write

   $$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \qquad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

   Then

   $$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

   and

   $$\begin{aligned} (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) &= (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

   so $A + B \in V$.

   (b) **Closed under scalar multiplication:** Let $A \in V$ and $\beta \in \mathbb{R}$ and write

   $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   Then

   $$\beta A = \begin{bmatrix} \beta a & \beta b \\ \beta c & \beta d \end{bmatrix}$$

   and

   $$\begin{aligned} (\beta a) + (\beta b) + (\beta c) &= \beta(a + b + c) \\ &= \beta(0) \\ &= 0, \end{aligned}$$

   so $\beta A \in V$.

   (c) **Contains the zero vector:** $0 + 0 + 0 = 0$, so

   $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V.$$

2. Find a basis for V. Name it B_1. What is the dimension of V?

Solution: There are multiple correct solutions to this problem. Your choice of basis may be slightly different.

First rewrite V:

$$\begin{aligned} V &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a + b + c = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a = -b - c \right\} \\ &= \left\{ \begin{bmatrix} -b-c & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{R} \right\} \qquad (*) \end{aligned}$$

We can find basis elements by choosing values for $b, c, d \in \mathbb{R}$.

- $b = 1,\ c = d = 0$: $\quad A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$
- $c = 1,\ b = d = 0$: $\quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$
- $d = 1,\ b = c = 0$: $\quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

We claim that $\boxed{B_1 = \{A_1, A_2, A_3\}}$ is a basis for V. There are three properties to show:

(i) $B_1 \subset V$: Each matrix A_i was determined using form in the rewritten definition $(*)$ for V. Therefore each A_i is an element of V.

(ii) $V = \operatorname{span}(B_1)$: Every element of V can be written as

$$\begin{aligned} \begin{bmatrix} -b-c & b \\ c & d \end{bmatrix} &= \begin{bmatrix} -b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -c & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \\ &= bA_1 + cA_2 + dA_3 \end{aligned}$$

for some $b, c, d \in \mathbb{R}$. That is, every element of V can be written as a linear combination of elements in B_1. Since $B_1 \subset V$ we get $V = \operatorname{span}(B_1)$.

(iii) B_1 is linearly independent: Let $b, c, d \in \mathbb{R}$ and set a linear combination equal to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$:

$$\begin{gathered} bA_1 + cA_2 + dA_3 = \mathbf{0}_{2\times 2} \\ \downarrow \\ \begin{bmatrix} -b-c & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{gathered}$$

From the matrix coordinates we find $b = c = d = 0$, so B_1 is linearly independent.

Since all three properties are satisfied, B_1 is a basis for V. Since B_1 has exactly 3 elements, $\boxed{\text{the dimension of } V \text{ is } 3.}$

3. Show that W is a vector space over $\mathbb{R}$.

Solution:

We will show that W is a subspace of $\mathbb{R}^{\mathbb{R}}$ (the space of all functions from $\mathbb{R}$ to $\mathbb{R}$). Check the 3 subspace axioms

(a) Closed under addition: Let $g(x), h(x) \in W$ and write

$$g(x) = \alpha_1 x + \beta_1 x^2 + \gamma_1 e^x, \qquad h(x) = \alpha_2 x + \beta_2 x^2 + \gamma_2 e^x,$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ for $i = 1, 2$. Then

$$g(x) + h(x) = (\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)x^2 + (\gamma_1 + \gamma_2)e^x$$

and

$$\begin{aligned}(\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) + (\gamma_1 + \gamma_2) &= (\alpha_1 + \beta_1 + \gamma_1) + (\alpha_2 + \beta_2 + \gamma_2) \\ &= 0 + 0 \\ &= 0,\end{aligned}$$

so $g(x) + h(x) \in W$.

(b) Closed under scalar multiplication: Let $g(x) \in W$ and $k \in \mathbb{R}$ and write

$$g(x) = \alpha x + \beta x^2 + \gamma e^x$$

Then

$$kg(x) = k\alpha x + k\beta x^2 + k\gamma e^x$$

and

$$\begin{aligned}k\alpha + k\beta + k\gamma &= k(\alpha + \beta + \gamma) \\ &= k(0) \\ &= 0,\end{aligned}$$

so $kg(x) \in W$.

(c) W contains the zero vector: $0 + 0 + 0 = 0$, so

$$\mathbf{0} = 0(x) + 0(x^2) + 0(e^x) \in W.$$

4. Find a basis for W. Name it B_2. What is the dimension of W?

Solution: There are multiple correct solutions to this problem. Your choice of basis may be slightly different.

First rewrite W:

$$\begin{aligned} W &= \{\alpha x + \beta x^2 + \gamma e^x : \alpha, \beta, \gamma \in \mathbb{R},\ \alpha + \beta + \gamma = 0\} \\ &= \{\alpha x + \beta x^2 + \gamma e^x : \alpha, \beta, \gamma \in \mathbb{R},\ \gamma = -\alpha - \beta\} \\ &= \{\alpha x + \beta x^2 + (-\alpha - \beta) e^x : \alpha, \beta, \gamma \in \mathbb{R}\} \qquad (**) \end{aligned}$$

We can find basis elements by choosing values for $\alpha, \beta \in \mathbb{R}$.

- $\alpha = 1,\ \beta = 0$: $\quad g_1(x) = x - e^x$
- $\alpha = 0,\ \beta = 1$: $\quad g_2(x) = x^2 - e^x$

We claim that $\boxed{B_2 = \{g_1, g_2\}}$ is a basis for W.

There are three properties to show:

(i) $B_2 \subset W$: Each function g_i was determined using form in the rewritten definition $(**)$ for W. Therefore each g_i is an element of W.

(ii) $W = \text{span}(B_2)$: Every element of W can be written as

$$\begin{aligned} \alpha x + \beta x^2 + (-\alpha - \beta) e^x &= \alpha(x - e^x) + \beta(x^2 - e^x) \\ &= \alpha g_1(x) + \beta g_2(x) \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$. That is, every element of W can be written as a linear combination of elements in B_2. Since $B_2 \subset W$ we get $W = \text{span}(B_2)$.

(iii) B_2 is linearly independent: Let $\alpha, \beta \in \mathbb{R}$ and set a linear combination equal to 0:

$$\begin{aligned} \alpha g_1(x) + \beta g_2(x) &= 0 \qquad \text{for all } x \in \mathbb{R} \\ &\downarrow \\ \alpha x + \beta x^2 + (-\alpha - \beta) e^x &= 0 \qquad \text{for all } x \in \mathbb{R} \end{aligned}$$

Plug in $x = 0$ to get

$$(-\alpha - \beta) = 0$$

Now that we know $(-\alpha - \beta) = 0$, the equation becomes

$$\alpha x + \beta x^2 = 0 \qquad \text{for all } x \in \mathbb{R}.$$

Plug in $x = 1$ and $x = -1$ to get

$$\begin{aligned} \alpha + \beta &= 0 \\ -\alpha + \beta &= 0 \end{aligned}$$

Adding these equations together gives $2\beta = 0$, so $\beta = 0$. Then the first equation becomes

$$\alpha + 0 = 0 \qquad \rightarrow \qquad \alpha = 0.$$

Thus $\alpha = \beta = 0$, so B_2 is linearly independent.

Since all three properties are satisfied, B_2 is a basis for W. Since B_2 has exactly 2 elements, $\boxed{\text{the dimension of } W \text{ is } 2.}$

5. Consider the function $f \colon V \to W$ given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax + bx^2 + ce^x.$$

Show that f is a linear map.

Solution: We must show that f preserves addition and scalar multiplication. That is, for $A, B \in V$ and $k \in \mathbb{R}$ we must show

- $f(A+B) = f(A) + f(B)$
- $f(kA) = kf(A)$

***f* preserves addition:** Let $A, B \in V$ and write

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \qquad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

Then

$$\begin{aligned} f(A+B) &= f\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right) \\ &= (a_1 + a_2)x + (b_1 + b_2)x^2 + (c_1 + c_2)e^x \end{aligned}$$

and

$$\begin{aligned} f(A) + f(B) &= f\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= (a_1x + b_1x^2 + c_1e^x) + (a_2x + b_2x^2 + c_2e^x) \\ &= (a_1 + a_2)x + (b_1 + b_2)x^2 + (c_1 + c_2)e^x \end{aligned}$$

so $f(A+B) = f(A) + f(B)$.

***f* preserves scalar multiplication:** Let $k \in \mathbb{R}$ and $A \in V$ and write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\begin{aligned} f(kA) &= f\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) \\ &= kax + kbx^2 + kce^x \\ &= k(ax + bx^2 + ce^x) \\ &= kf(A). \end{aligned}$$

6. Find the matrix $[f]_{B_2,B_1}$.

Solution: Your solution depends on your choice of bases in problems 2 and 4. If you chose different bases than the ones I chose, then your answer will probably look different.

To find the matrix $[f]_{B_2,B_1}$ we must apply f to each element of B_1 and rewrite the result in terms of B_2. Recall that I chose

- $B_1 = \{A_1, A_2, A_3\} = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
- $B_2 = \{g_1, g_2\} = \{x - e^x, x^2 - e^x\}$

To find the first column of $[f]_{B_2,B_1}$ apply f to A_1 and write result in terms of B_2.

$$\begin{aligned} f(A_1) &= f\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= -x + x^2 \\ &= -(x - e^x) + (x^2 - e^x) \\ &= (-1)g_1 + (1)g_2 \end{aligned}$$

The coefficients on g_1 and g_2 tell us that the first column of $[f]_{B_2,B_1}$ is

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now apply this process to A_2 and A_3:

$$\begin{aligned} f(A_2) &= -x + e^x \\ &= -(x - e^x) \\ &= (-1)g_1 + (0)g_2 \end{aligned}$$

and

$$\begin{aligned} f(A_3) &= 0 \\ &= (0)g_1 + (0)g_2 \end{aligned}$$

so the second and third columns of $[f]_{B_2,B_1}$ are

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

respectively. Therefore

$$\boxed{[f]_{B_2,B_1} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.}$$

7. Let V be the set of all continuous functions from the interval $[0,1]$ to $\mathbb{R}$. Consider the map $F\colon V \to \mathbb{R}$ given by

$$F(u) = \int_0^1 x^2 u(x)\,dx \qquad \forall u \in V.$$

Is F linear? Verify your answer.

Solution: Yes, F is linear. To prove this, we need to show that F preserves addition and scalar multiplication.

Recall the definitions of function addition and scalar multiplication:

$$\begin{aligned} (u+v)(x) &= u(x) + v(x) && \text{for } u, v \in V \\ (\alpha u)(x) &= \alpha u(x) && \text{for } u \in V \text{ and } \alpha \in \mathbb{R} \end{aligned}$$

Now we check the properties of a linear function:

***F* preserves addition:** Let $u, v \in V$. Then

$$\begin{aligned} F(u+v) &= \int_0^1 x^2 (u+v)(x)\,dx \\ &= \int_0^1 x^2 (u(x) + v(x))\,dx \\ &= \int_0^1 x^2 u(x) + x^2 v(x)\,dx \\ &= \int_0^1 x^2 u(x)\,dx + \int_0^1 x^2 v(x)\,dx && \text{(by linearity of integration)} \\ &= F(u) + F(v). \end{aligned}$$

***F* preserves scalar multiplication:** Let $\alpha \in \mathbb{R}$ and $u \in V$. Then

$$\begin{aligned} F(\alpha u) &= \int_0^1 x^2 (\alpha u)(x)\,dx \\ &= \int_0^1 x^2 (\alpha u(x))\,dx \\ &= \alpha \int_0^1 x^2 u(x)\,dx && \text{(by linearity of integration)} \\ &= \alpha F(u). \end{aligned}$$