Homework 4
Answer Key

For problems 1, 2, 3, let V be the set of all 3×2 matrices with real coefficients such that the sum of the entries in each row is equal to 0.

1. Show that V is a vector space over $\mathbb{R}$.

   Solution: Since V is a subset of the vector space $M_{3\times 2}(\mathbb{R})$, we only need to show that it is also a subspace of $M_{3\times 2}(\mathbb{R})$.

   - $\mathbf{0} \in \boldsymbol{V}$**:** Notice that the sum of the coordinates in each row of
   $$\mathbf{0}_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
   is $0 + 0 = 0$, so $\mathbf{0}_{3\times 2} \in V$.
   - **Closed under addition:** Let $A, B \in V$ and write
   $$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \\ e_1 & f_1 \end{bmatrix}, \qquad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \\ e_2 & f_2 \end{bmatrix}$$
   Then
   $$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \\ e_1 + e_2 & f_1 + f_2 \end{bmatrix}.$$
   The sum of the coordinates in the first row of $A + B$ is
   $$a_1 + a_2 + b_1 + b_2 = (a_1 + b_1) + (a_2 + b_2) = 0 + 0 = 0.$$
   The sums in the other rows are also 0 by similar reasoning. Therefore $A + B \in V$
   - **Closed under scalar multiplication:** Let $A \in V$ and $\lambda \in \mathbb{R}$ and write
   $$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$
   Then
   $$\lambda A = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \\ \lambda e & \lambda f \end{bmatrix}$$
   The sum of the coordinates in the first row of $A + B$ is
   $$\lambda a + \lambda b = \lambda(a + b) = \lambda(0) = 0.$$
   The sums in the other rows are also 0 by similar reasoning. Therefore $\lambda A \in V$

   Thus V satisfies all of the subspace axioms, so V is a subspace of $M_{3\times 2}(\mathbb{R})$ and hence a vector space.

2. Find a basis of V. Call it $\mathcal{B}$. What is $\dim_{\mathbb{R}} V$?

Solution: We can write the set V as

$$\begin{aligned} V &= \left\{ \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} : a,b,c,d,e,f \in \mathbb{R},\ a+b=0,\ c+d=0,\ e+f=0 \right\} \\ &= \left\{ \begin{bmatrix} a & -a \\ c & -c \\ e & -e \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \end{aligned}$$

Let

$$\boxed{\mathcal{B} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}}$$

and notice that $\mathcal{B} \subset V$. Now every element of V can be written as a linear combination

$$a \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}, \qquad a,c,e \in \mathbb{R}$$

of elements of $\mathcal{B}$. It is easy to check that $\mathcal{B}$ is linearly independent. Set a linear combination equal to 0:

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} &= a \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}, \qquad a,c,e \in \mathbb{R} \\ &= \begin{bmatrix} a & -a \\ c & -c \\ e & -e \end{bmatrix}. \end{aligned}$$

The coordinates in the first column give $a = c = e = 0$, so $\mathcal{B}$ is linearly independent. The dimension of V is $\boxed{\dim_{\mathbb{R}}(V) = |\mathcal{B}| = 3}$.

3. Find the coordinate vector (in basis $\mathcal{B}$) of the following matrix:

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 3 \\ 0 & 0 \end{bmatrix}$$

Solution: To find the $\mathcal{B}$-coordinate vector of A, we must first write A in terms of the elements of $\mathcal{B}$:

$$\begin{bmatrix} 2 & -2 \\ -3 & 3 \\ 0 & 0 \end{bmatrix} = (2) \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} + (0) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

The coeffiecients 2, -3, and 0 are the components of the coordinate vector:

$$\boxed{[A]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.}$$

For problems 4, 5, 6, 7, let $f\colon M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ be a function defined as

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a(x-1)^2 + bx$$

Here $P_2(\mathbb{R})$ denotes the vector space of all polynomials with real coefficients of degree ≤ 2.

4. Show that f is a linear map.

   Solution: we need to check

   (a) $f(A+B) = f(A) + f(B)$,

   (b) $f(\lambda A = \lambda f(A)$

   for all $A, B \in M_{2\times 2}(\mathbb{R})$ and all $\lambda \in \mathbb{R}$. Write

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

   Then

   (a)

$$\begin{aligned} F(A+B) &= F\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= (a_1+a_2)(x-1)^2 + (b_1+b_2)x \\ &= a_1(x-1)^2 + b_1x + a_2(x-1)^2 + b_2x \\ &= F(A) + F(B), \end{aligned}$$

   (b)

$$\begin{aligned} F(\lambda A) &= F\left(\begin{bmatrix} \lambda a_1 & \lambda b_1 \\ \lambda c_1 & \lambda d_1 \end{bmatrix}\right) \\ &= (\lambda a_1)(x-1)^2 + (\lambda b_1)x \\ &= \lambda(a_1(x-1)^2 + b_1x) \\ &= \lambda F(A). \end{aligned}$$

5. Find a matrix representation of f.

Solution: To do this, we first need to choose bases for $M_{2\times 2}(\mathbb{R})$ and $P_2(\mathbb{R})$. The standard bases are

$$\begin{aligned}
\mathcal{B}_1 &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \{E_1, E_2, E_3, E_4\} \qquad && \text{for } M_{2\times 2}(\mathbb{R}) \\
\mathcal{B}_2 &= \{1, x, x^2\} && \text{for } P_2(\mathbb{R}).
\end{aligned}$$

Now the columns of the matrix are the $\mathcal{B}_2$-coordinate vectors of $f(E_i)$ for $i = 1, 2, 3, 4$. To find these coordinate vectors, we must write $f(E_i)$ in terms of $\mathcal{B}_2$:

$$\begin{aligned}
f(E_1) &= f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&= (1)(x-1)^2 \\
&= 1 - 2x + x^2 && \to \qquad [f(E_1)]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
f(E_2) &= (1)(x) && \to \qquad [f(E_2)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
f(E_3) &= 0 && \to \qquad [f(E_3)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
f(E_4) &= 0 && \to \qquad [f(E_4)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Therefore the matrix representation of f is

$$\boxed{[f]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.}$$

6. Find a basis of null(f). What is the nullity of f?

Solution: The null space of f is the set of all elements $A \in M_{2\times 2}(\mathbb{R})$ such that $F(A) = 0$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an arbitrary element in $M_{2\times 2}(\mathbb{R})$. Then

$$\begin{aligned} f(A) &= a(x-1)^2 + bx \\ &= a + (-2a+b)x + ax^2. \end{aligned}$$

Now $f(A) = a + (-2a+b)x + ax^2$ is zero for all $x \in \mathbb{R}$ if and only if each coefficient is 0:

$$\begin{aligned} a &= 0 \\ -2a + b &= 0 \\ a &= 0 \end{aligned}$$

Since $a = 0$ by the first equation, we must have $b = 0$ from the second equation. Therefore $f(A) = 0$ if and only if $a = b = 0$. We can write the null space as

$$\text{null}(f) = \left\{ \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} : c, d \in \mathbb{R} \right\}.$$

It is easy to check that

$$\boxed{\mathcal{B} = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}$$

is a basis for null(f). The nullity of f is

$$\boxed{\dim(\text{null}(f)) = |\mathcal{B}| = 2.}$$

7. Find a basis of range(f). What is the rank of f?

Soltuion: We can determine the rank of f using the rank-nullity theorem:

$$\begin{gathered} \text{rank}(f) + \text{nullity}(f) = \dim(M_{2\times 2}(\mathbb{R})) \\ \downarrow \\ \text{rank}(f) + 2 = 4 \\ \downarrow \\ \boxed{\text{rank}(f) = 2} \end{gathered}$$

The range of f is

$$\begin{aligned} \text{range}(f) &= \{f(A) : A \in M_{2\times 2}(\mathbb{R})\} \\ &= \{a(x-1)^2 + b : a, b \in \mathbb{R}\} \end{aligned}$$

It is easy to see from this that the set $\boxed{\mathcal{B} = \{(x-1)^2, x\}}$ spans range(f). Now range(f) is a 2-dimensional space (since $\dim(\text{range}(f)) = \text{rank}(f) = 2$) and $\mathcal{B}$ is a set of exactly 2 elements that spans range(f), so $\mathcal{B}$ must be a basis for range(f).