Homework 8

Answer Key

1. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map given by $f(v) = Av$ where

$$A = \begin{bmatrix} 3 & -2 & -2 \\ -1 & 4 & 2 \\ 2 & -4 & -2 \end{bmatrix}.$$

Is f diagonalizable? If it is, express $V = \mathbb{R}^3$ as a direct sum of one-dimensional invariant subspaces under f; then find a basis of V in which f is represented by a diagonal matrix.

Solution: First find the characteristic polynomial of A:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & -2 & -2 \\ -1 & 4-\lambda & 2 \\ 2 & -4 & -2-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -4 & -2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -2 & -2 \\ -4 & -2-\lambda \end{vmatrix} + (2)\begin{vmatrix} -2 & -2 \\ 4-\lambda & 2 \end{vmatrix} \\
&= (3-\lambda)[(4-\lambda)(-2-\lambda) - (2)(-4)] \\
&\quad + [(-2)(-2-\lambda) - (-2)(-4)] \\
&\quad + (2)[(-2)(2) - (-2)(4-\lambda)] \\
&= (3-\lambda)(\lambda^2 - 2\lambda) + (2\lambda - 4) + (2)(-2\lambda + 4) \\
&= (3-\lambda)(\lambda)(\lambda - 2) + 2(\lambda - 2) - 4(\lambda - 2) \\
&= (\lambda - 2)[(3-\lambda)(\lambda) + 2 - 4] \\
&= (\lambda - 2)(-1)(\lambda^2 - 3\lambda + 2) \\
&= -(\lambda - 2)(\lambda - 2)(\lambda - 2) \\
&= -(\lambda - 2)^2(\lambda - 1).
\end{aligned}$$

Setting $-(\lambda-2)^2(\lambda-1) = 0$ gives eigenvalues $\lambda = 1$ and $\lambda = 2$.

To see if f is diagonalizable, we need to check if the direct sum of the eigenspaces is all of V. That is, we need to check if the dimensions of the eigenspaces add up to $\dim(V) = 4$.

We now need to find the eigenvectors of A. This means solving the equation

$$(A - \lambda I)v = 0$$

where $v = (v_1, v_2, v_3) \in V = \mathbb{R}^3$.

First let $\lambda = 1$. The augmented form of the above equation is

$$\left[\begin{array}{ccc|c} 3-1 & -2 & -2 & 0 \\ -1 & 4-1 & 2 & 0 \\ 2 & -4 & -2-1 & 0 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & -4 & -3 & 0 \end{array}\right].$$

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Now use row reduction:

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & -4 & -3 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & -4 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_1 \to R_2 \text{ and } R_3-2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+R_2 \to R_1 \text{ and } R_3+R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives the equations

$$\begin{aligned} v_1 + v_2 &= 0 \\ 2v_2 + v_3 &= 0 \end{aligned}$$

so $v_1 = -v_2$ and $v_3 = -2v_2$. The eigenvectors v can be written as

$$v = \begin{bmatrix} -v_2 \\ v_2 \\ -2v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

where $v_2 \in \mathbb{R}$. Therefore, a basis for the eigenspace E_1 is

$$B_1 = \{(-1, 1, -2)\}$$

Now let $\lambda = 2$. We follow the same process of solving $(A - \lambda I)v = 0$:

$$\left[\begin{array}{ccc|c} 3-2 & -2 & -2 & 0 \\ -1 & 4-2 & 2 & 0 \\ 2 & -4 & -2-2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ -1 & 2 & 2 & 0 \\ 2 & -4 & -4 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so we get the single equation

$$v_1 - 2v_2 - 2v_3 = 0.$$

This means that $v_1 = 2v_2 + 2v_3$, so the eigenvectors v can be written as

$$v = \begin{bmatrix} 2v_2 + 2v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

where $v_2, v_3 \in \mathbb{R}$. A basis for E_2 is

$$B_2 = \{(2, 1, 0), (2, 0, 1)\}.$$

Since $\dim(E_1) + \dim(E_2) = 1 + 2 = 3 = \dim(V)$ the linear map f is diagonalizable. Under the basis

$$B = B_1 \cup B_2 = \{(-1, 1, 2), (2, 1, 0), (2, 0, 1)\}$$

the map f is represented by the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

2. Let $f\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear map given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 2a+b \\ 2c & c+2d \end{bmatrix}.$$

Is f diagonalizable? If it is, express $V = M_{2\times 2}(\mathbb{R})$ as a direct sum of one-dimensional invariant subspaces under f; then find a basis of V in which f is represented by a diagonal matrix.

Solution: Since f is a map from $M_{2\times 2}(\mathbb{R})$ to $M_{2\times 2}(\mathbb{R})$ we begin by finding a basis for $M_{2\times 2}(\mathbb{R})$ and representing f as a 4×4 matrix. Consider the basis B for $M_{2\times 2}(\mathbb{R})$ given by

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Applying f to the first element of B, we get

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so the first row of the matrix for f is

$$\left[f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{B,B} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Continuing in this way, we find that

$$A = [f]_{B,B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Since this is a triangular matrix, the eigenvalues are simply the entries on the diagonal. That is, the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

To see if f is diagonalizable, we need to check if the direct sum of the eigenspaces is all of V. That is, we need to check if the dimensions of the eigenspaces add up to $\dim(V) = 4$.

We must calculate the eigenspace for each eigenvector. We want to solve

$$(A - \lambda I)v = 0$$

for $v = (v_1, v_2, v_3, v_4) \in V$. First, let $\lambda = 1$. The augmented form of the above equation is

$$\left[\begin{array}{cccc|c} 1-1 & 0 & 0 & 0 & 0 \\ 2 & 1-1 & 0 & 0 & 0 \\ 0 & 0 & 2-1 & 0 & 0 \\ 0 & 0 & 1 & 2-1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so $v_1 = v_3 = v_4 = 0$ and the eigenvectors v can be written as

$$v = \begin{bmatrix} 0 \\ v_2 \\ 0 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

where $v_2 \in \mathbb{R}$. A basis for the eigenspace E_1 is $B_1 = \{(0,1,0,0)\}$ so $\dim(E_1) = 1$.

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Now let $\lambda = 2$. We follow the same process of solving $(A - \lambda I)v = 0$:

$$\left[\begin{array}{cccc|c} 1-2 & 0 & 0 & 0 & 0 \\ 2 & 1-2 & 0 & 0 & 0 \\ 0 & 0 & 2-2 & 0 & 0 \\ 0 & 0 & 1 & 2-2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so $v_1 = v_2 = v_3 = 0$ and the eigenvector v can be written as

$$v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_4 \end{bmatrix} = v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $v_4 \in \mathbb{R}$. A basis for the eigenspace E_2 is $B_2 = \{(0,0,0,1)\}$ so $\dim(E_2) = 1$.

Since

$$\dim(E_1) + \dim(E_2) = 1 + 1 = 2 \neq 4 = \dim(V)$$

the linear map f is not diagonalizable.

3. Let V be the vector space of all smooth (infinitely differentiable) functions from $\mathbb{R}$ to $\mathbb{R}$. Let $F\colon V \to V$ be a linear map defined by $F(u) = u'$. Find all eigenvectors and eigenvalues of F.

Solution: Since V is an infinite-dimensional vector space, we cannot solve this problem by converting to a matrix equation. Instead, we need to find the eigenvectors directly. Recall that an eigenvector of F is a vector $v \in V$ such that

$$F(v) = \lambda v$$

for some λ in the base field $F = \mathbb{R}$. In this case, we get the equation

$$v' = \lambda v.$$

where $v = v(t)$ is a (smooth) function of the variable t. This is a differential equation that can be solved using separation of variables:

$$\begin{gathered} v' = \lambda v. \\ \downarrow \\ \frac{1}{v} v' = \lambda \\ \downarrow \\ \int \frac{1}{v} v' dt = \int \lambda dt \\ \downarrow \\ \ln(v) = \lambda t + C \\ \downarrow \\ v = e^{\lambda t + C} = De^{\lambda t} \end{gathered}$$

In this case there was no restriction on λ when solving for v above, so every element of $\mathbb{R}$ is an eigenvalue of F. The eigenvectors associated to the eigenvalue λ are

$$\{De^{\lambda t} : D \in \mathbb{R}\}$$

A basis for the eigenspace E_λ is $B_\lambda = \{e^{\lambda t}\}$, so each eigenspace has dimension 1.

4. Let $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a linear map given by $f(v) = Av$ where

$$A = \begin{bmatrix} 1-i & 2-i \\ 0 & 2+i \end{bmatrix}.$$

Is f diagonalizable? If it is, express $V = \mathbb{C}^2$ as a direct sum of one-dimensional invariant subspaces under f; then find a basis of V in which f is represented by a diagonal matrix.

Solution: Notice that A is a triangular matrix, so the eigenvalues are the entries on the diagonal:

$$\lambda = 1-i \qquad \text{or} \qquad \lambda = 2+i.$$

Since V has dimension 2 and there are 2 distinct eigenvalues, we already know that f is diagonalizable. We now need to find the eigenspace for each eigenvalue. That is, we need to solve

$$(A - \lambda I)v = 0$$

for $v = (v_1, v_2) \in V = \mathbb{C}^2$. First let $\lambda = 1-i$:

$$\left[\begin{array}{cc|c} 1-i-(1-i) & 2-i & 0 \\ 0 & 2+i-(1-i) & 0 \end{array}\right] \quad \to \quad \left[\begin{array}{cc|c} 0 & 2-i & 0 \\ 0 & 1+2i & 0 \end{array}\right].$$

Perform row reduction:

$$\begin{aligned} \left[\begin{array}{cc|c} 0 & 2-i & 0 \\ 0 & 1+2i & 0 \end{array}\right] & \xrightarrow{\frac{1}{2-i}R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1+2i & 0 \end{array}\right] \\ & \xrightarrow{R_2-(1+2i)R_1 \to R_2} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]. \end{aligned}$$

This gives $v_2 = 0$, so the eigenvectors v are

$$v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $v_1 \in \mathbb{C}$. A basis for the eigenspace E_{1-i} is $\{(1,0)\}$.

Now let $\lambda = 2$. We follow the same process of solving $(A - \lambda I)v = 0$:

$$\left[\begin{array}{cc|c} 1-i-(2+i) & 2-i & 0 \\ 0 & 2+i-(2+i) & 0 \end{array}\right] \quad \to \quad \left[\begin{array}{cc|c} -1-2i & 2-i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Perform row reduction:

$$\left[\begin{array}{cc|c} -1-2i & 2-i & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{-1-2i}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & \frac{2-i}{-1-2i} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

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The fraction in the reduced matrix can be simplified:

$$\begin{aligned}
\frac{2-i}{-1-2i} &= \frac{2-i}{-1-2i} \cdot \frac{-1+2i}{-1+2i} \\
&= \frac{(2-i)(-1+2i)}{(-1)^2-(2i)^2} \\
&= \frac{-2+4i+i+2}{1+4} \\
&= \frac{5i}{5} \\
&= i,
\end{aligned}$$

so the reduced matrix is

$$\left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

This gives $v_1 + iv_2 = 0$, so $v_1 = -iv_2$. The eigenvectors v can be written as

$$v = \begin{bmatrix} -iv_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

where $v_2 \in \mathbb{C}$. A basis for the eigenspace E_{2+i} is $\{(-i, 1)\}$.

As we noted earlier, f is diagonalizable. Under the basis

$$B = \{(1, 0), (-i, 1)\}$$

the map f is represented by the diagonal matrix

$$\begin{bmatrix} 1-i & 0 \\ 0 & 2+i \end{bmatrix}.$$