Lecture 3 (10/23/2019)

$f: V \to W$ linear.

f is called
- monomorphism (linear + injective) if $\mathrm{null}(f) = \{0\}$.
- epimorphism (linear + surjective) if $\mathrm{range}(f) = W$.
- isomorphism (linear + bijective) if it is both monomorphism and epimorphism.

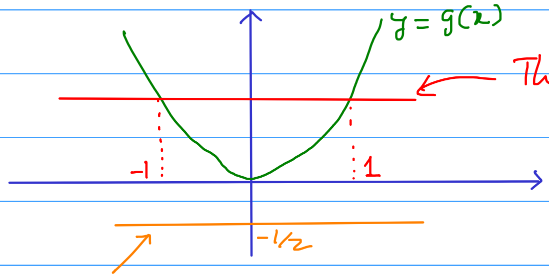
mono : one, single
epi : on, over
iso : equal

Greek roots

"injective" is often referred to as "one-to-one", meaning different inputs give different outputs.

"surjective" is often refered to as "onto", meaning every vector $w \in W$ comes from some element $v \in V$, i.e. $f(v) = w$.

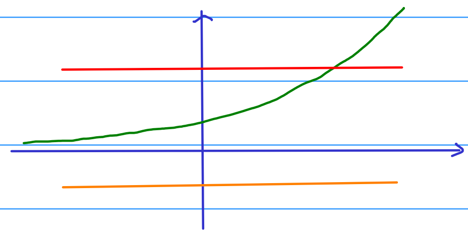
Ex: The map $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is not one-to-one, not onto, and not linear.



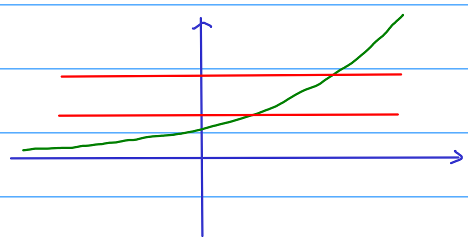
This horizontal line intersects the graph of g at two points $(-1,1)$ and $(1,1)$. Since $g(-1) = g(1)$, g is not one-to-one.

This horizontal line doesn't intersect the graph of g. The point $-1/2$ in the target set $\mathbb{R}$ doesn't come from any point in the domain $\mathbb{R}$. That is, there is no point $x \in \mathbb{R}$ such that $g(x) = x^2 = -1/2$.

Ex: the map $h: \mathbb{R} \to \mathbb{R}$, $h(x) = e^x$
is one-to-one, but not onto.



Ex: the map $k: \mathbb{R} \to (0, \infty)$, $k(x) = e^x$
is both one-to-one and onto.



Note: a map has three components:

- domain
- target set
- law of mapping

We see that functions h and k in the previous examples have the same domain and law of mapping. The target sets make them different. Function k is onto, but h is not.

* Theorem:

Let $f: V \to W$ be a linear map. Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V. Then

(i) f is monomorphic iff $\{f(v_1), \dots, f(v_n)\}$ is linearly ind.

(ii) f is epimorphic iff $\{f(v_1), \dots, f(v_n)\}$ spans W.

(iii) f is isomorphic iff $\{f(v_1), \dots, f(v_n)\}$ is a basis of W.

A monomorphism preserves linear independence: the images of linearly independent vectors are also linearly independent.

An epimorphism preserves spanning sets: the image of a spanning set of V is a spanning set of W.

An isomorphism preserves bases: the image of a basis of V is a basis of W.

Ex: Consider the function

$$f: \mathbb{R}^4 \longrightarrow M_{1\times 2}(\mathbb{R})$$

$$f(a,b,c,d) = [a+b \quad c+d]$$

Is f a monomorphism, epimorphism or isomorphism?

Let's pick a basis for $\mathbb{R}^4$, say the standard basis

$$B_0 = \{e_1, e_2, e_3, e_4\}.$$

We have

$$f(e_1) = f(1,0,0,0) = \underbrace{[1 \quad 0]}_{w_1}$$

$$f(e_2) = f(0,1,0,0) = \underbrace{[1 \quad 0]}_{w_2}$$

$$f(e_3) = f(0,0,1,0) = \dots = w_3$$

$$f(e_4) = f(0,0,0,1) = \dots = w_4$$

Because $w_1 = w_2$, the set $\{w_1, w_2, w_3, w_4\}$ is linearly dependent.

Indeed, the following linear combination gives 0:

$$1w_1 + (-1)w_2 + 0w_3 + 0w_4 = 0.$$

Therefore, f is not monomorphic.

* General result:

If $f: V \to W$ is linear and $\dim V > \dim W$ then f is never one-to-one.

This is a nice consequence of the rank-nullity theorem. We will discuss more on this result next time.