MATH 342, MIDTERM EXAM, FALL 2019

Name	Recitation time	Student ID

- Answer to each problem must be written coherently in full sentences. Answers not supported by valid arguments will not receive full credit. Do not use ambiguous symbols such as $\rightarrow$, ?, $\ldots$, $\therefore$ Instead, use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

- Read carefully the description of each problem. Make sure that you do all parts of the problem.

- Doing correctly Problems 1,2,3,4,5 will result in 100% credit of the exam. You can earn extra credit by doing Problem 6.

Problem	Possible points	Earned points
1	10	
2	10	
3	10	
4	10	
5	10	
6	5	
Total	55	

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Problem 1. (10 points) Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_2 = x_1 + x_3\}$. Show that V is a vector space over $\mathbb{R}$.

We see that $V \subset \mathbb{R}^3$, and that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Thus, we only need to check 3 things:

(1) 0 belongs to V,

(2) V is closed under addition

(3) V is closed under scaling.

Check (1):

Vector $(0,0,0)$ indeed belongs to V because $\underbrace{0}_{x_2} = \underbrace{0}_{x_1} + \underbrace{0}_{x_3}$.

Check (2): Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two elements of V.

We want to show that $x+y \in V$. We have

$$x+y = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3).$$

To show $x+y \in V$ is to show that

$$x_2+y_2 = (x_1+y_1) + (x_3+y_3).$$

$$RHS = (x_1+x_3) + (y_1+y_3) = x_2 + y_2 \quad (\text{because } x, y \in V).$$
$$= LHS.$$

We have showed that (2) is true.

Check 3: Let $x = (x_1, x_2, x_3) \in V$ and $c \in \mathbb{R}$. We want to show that $cx \in V$. We have $cx = (cx_1, cx_2, cx_3)$. To show that $cx \in V$ is to show that $cx_2 = cx_1 + cx_3$.

$$RHS = c(x_1 + x_3) = cx_2 \quad (\text{because } x \in V)$$
$$= LHS.$$ We have showed that (3) is true.

Problem 2. (10 points) Let V be the vector space given in Problem 1. Find a basis of V. Determine the dimension of V.

We can write
$$V = \{(x_1, x_2, x_3) : x_2 = x_1 + x_3\}$$
$$= \{(x_1, x_1 + x_3, x_3) : x_1, x_3 \in \mathbb{R}\}$$
$$= \{x_1(1,1,0) + x_3(0,1,1) : x_1, x_3 \in \mathbb{R}\}$$
$$= \text{span}\{\underbrace{(1,1,0)}_{v_1}, \underbrace{(0,1,1)}_{v_2}\}$$

The set $B = \{v_1, v_2\}$ spans V. We will now show that B is linearly independent. Consider $a, b \in \mathbb{R}$ such that $av_1 + bv_2 = 0$.

We have
$$av_1 + bv_2 = a(1,1,0) + b(0,1,1) = (a, a+b, b).$$

This vector is equal to $(0,0,0)$ only if $a = b = 0$. Therefore, B is linearly independent. Thus, B is a basis of V. Then

$$\dim V = \text{number of vectors in } B = 2.$$

Problem 3. (10 points) Consider a map $f : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ given by

$$f(a,b) = \begin{bmatrix} b & a+b \\ 0 & a \end{bmatrix} \qquad \forall\, a,b \in \mathbb{R}.$$

Show that f is a linear map.

We need to show 2 things:

(1) f is additive.

(2) f is scalar multiplicative.

Check (1):

Let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be in $\mathbb{R}^2$. Then

$$f(v_1 + v_2) = f(a_1 + a_2, b_1 + b_2) = \begin{bmatrix} b_1 + b_2 & a_1 + a_2 + b_1 + b_2 \\ 0 & a_1 + a_2 \end{bmatrix}$$

$$f(v_1) = \begin{bmatrix} b_1 & a_1 + b_1 \\ 0 & a_1 \end{bmatrix}$$

$$f(v_2) = \begin{bmatrix} b_2 & a_2 + b_2 \\ 0 & a_2 \end{bmatrix}$$

Then $f(v_1) + f(v_2) = \begin{bmatrix} b_1 & a_1 + b_1 \\ 0 & a_1 \end{bmatrix} + \begin{bmatrix} b_2 & a_2 + b_2 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 & a_1 + b_1 + a_2 + b_2 \\ 0 & a_1 + a_2 \end{bmatrix}$

Thus, $f(v_1) + f(v_2) = f(v_1 + v_2)$.

Check 2:

Let $v = (a, b) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. We have

$$f(cv) = f(ca, cb) = \begin{bmatrix} cb & ca + cb \\ 0 & ca \end{bmatrix},$$

$$f(v) = \begin{bmatrix} b & a + b \\ 0 & a \end{bmatrix},$$

$$c f(v) = \begin{bmatrix} cb & c(a+b) \\ 0 & ca \end{bmatrix} = \begin{bmatrix} cb & ca + cb \\ 0 & ca \end{bmatrix}.$$

Thus, $f(cv) = c f(v)$.

Problem 4. (10 points) Let f be the linear map given in Problem 3. Find a matrix representation of f.

We choose a basis for $\mathbb{R}^2$ as $\mathcal{B}_1 = \{\underbrace{(1,0)}_{e_1}, \underbrace{(0,1)}_{e_2}\}$,

a basis for $M_{2\times2}(\mathbb{R})$ as $\mathcal{B}_2 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$.

Then

$$[f]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{bmatrix} | & | \\ [f(e_1)]_{\mathcal{B}_2} & [f(e_2)]_{\mathcal{B}_2} \\ | & | \end{bmatrix}$$

We have $f(e_1) = f(1,0) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = E_2 + E_4$.

Thus, $[f(e_1)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Similarly, $f(e_2) = f(0,1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = E_1 + E_2$

Thus, $[f(e_2)]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Therefore, $[f]_{\mathcal{B}_2,\mathcal{B}_1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Problem 5. (10 points) Consider a linear map $F : P_2 \to P_1$ given by $F(u) = -u'$. Here P_n denotes the vector space of all polynomials with real coefficients of degree $\leq n$. Find a basis of null(F). What are the nullity and rank of F?

$$\text{null}(F) = \{ u \in P_2 : F(u) = 0 \}$$

$$= \{ u \in P_2 : -u' = 0 \}$$

$$= \{ u \in P_2 : u \text{ is a constant function} \}$$

$$= \text{span}\{1\}.$$

The null(F) has basis {1} and the nullity of F is therefore equal to $\underline{1}$.

By rank-nullity theorem,

$$\text{rank} + \underbrace{\text{nullity}}_{1} = \underbrace{\dim P_2}_{3}.$$

Hence, the rank of f is equal to $3-1=2$.

Problem 6. (5 points) Let F be the linear map given in Problem 5. Is F monomorphic, epimorphic, isomorphic or none of them? Verify your answer.

In this problem, $V = \mathbb{P}_2$ and $W = \mathbb{P}_1$.

$$\dim V = 3 > \dim W = 2.$$

Thus, F is neither monomorphic nor isomorphic.

Now we check if F is epimorphic.

$$\text{range}(F) = \{ F(u) : u \in P_2 \}$$

$$= \{ -u' : u \in P_2 \}$$

For each $u \in P_2$, we can write $u = ax^2 + bx + c$. Then

$$-u' = -(2ax + b) = -2ax - b.$$

Then range(F) can be rewritten as

$$\text{range}(F) = \{ -2ax - b : a, b \in \mathbb{R} \}$$

$$= \text{span}\{x, 1\}$$

Note that $\{x, 1\}$ is a basis of $\mathbb{P}_1$. Thus, $\text{range}(F) = \mathbb{P}_1$.

We conclude that F is epimorphic.