MTH 342 Worksheet 2

Week 1 – 10/03/2019

Name: **Answer Key** **Recitation time:** ________

Let F be a field of scalars and let V be a set with defined addition and scalar multiplication. The vector space axioms are

- (A0) Closed under addition: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

1. (A1) Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$.
2. (A2) Associativity of addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
3. (A3) Zero vector: There exists a vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
4. (A4) Additive inverse: For every vector $\mathbf{v} \in V$ there exists a vector $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

- (S0) Closed under scalar multiplication: If $\mathbf{v} \in V$ and $\alpha \in F$, then $\alpha\mathbf{v} \in V$.

5. (S1) Multiplicative identity: $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.
6. (S2) Associativity of scalar multiplication: $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$ for all $\mathbf{v} \in V$ and all scalars α, β.
7. (I1) Scalar distribution 1: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars α.
8. (I2) Scalar distribution 2: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ for all $\mathbf{v} \in V$ and all scalars α, β

1. Consider the set $S = \{(x, y) : x, y \in \mathbb{R} \text{ and } x \neq 0\} \cup \{(0, 0)\}$ with scalars in $\mathbb{R}$ and with addition and scalar multiplication defined componentwise. This is *not* a vector space. Why not? Which vector space axioms does S violate? Which does it satisfy?

Solution: First consider the axioms that S violates:

- S does not satisfy axiom A0 – it is not closed under addition. Let $\mathbf{v} = (1, 1)$ and $\mathbf{u} = (-1, 1)$. Then $\mathbf{u}, \mathbf{v} \in V$ since neither first coordinate is 0, but

$$\mathbf{u} + \mathbf{v} = (1, 1) + (-1, 1) = (0, 2) \notin V.$$

- I would say that axioms A1, A2, and I1 are violated (although this *might* be debatable). This is because they each assume that axiom A0 is true. For example, consider again $\mathbf{v} = (1, 0)$ and $\mathbf{u} = (-1, 1)$. Then the statement

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

of axiom A1 is meaningless, because $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$ do not exist (as elements of V). A similar problem occurs when checking axioms A2 and I1.

All other axioms are satisfied:

- A3: Consider the element $(0, 0) \in S$. Then $(0, 0) + (x, y) = (x, y)$ for any $(x, y) \in S$, so $\mathbf{0} = (0, 0)$ and axiom A3 is satisfied.
- A4: Let $(x, y) \in S$. If $x = 0$ then $y = 0$, and so $(0, 0) \in S$ is the additive inverse of (x, y). Otherwise $x \neq 0$, so $-x \neq 0$. Therefore $(-x, -y) \in S$. Now

$$(x, y) + (-x, -y) = (0 + 0) = \mathbf{0}$$

so $(-x, -y)$ is the additive inverse of (x, y) and axiom A4 is satisfied.

- S0: Let $\alpha \in \mathbb{R}$ and $(x, y) \in S$. There are three cases:
  (i) $\alpha = 0$: Then $\alpha(x, y) = (0, 0) \in S$.
  (ii) $\alpha \neq 0$, $x = 0$: Since $x = 0$, we also have $y = 0$, so $\alpha(x, y) = (0, 0) \in S$.
  (iii) $\alpha \neq 0$, $x \neq 0$: Then $\alpha x \neq 0$, so $\alpha(x, y) = (\alpha x, \alpha y) \in S$.
- S1: Let $(x, y) \in S$. Then $1(x, y) = ((1)x, (1)y) = (x, y)$.

- S2: Let $(x, y) \in S$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}(\alpha\beta)(x,y) &= ((\alpha\beta)x, (\alpha\beta)y) \\ &= (\alpha(\beta x), \alpha(\beta y)) \qquad \text{by associativity of multiplication in } \mathbb{R} \\ &= \alpha(\beta x, \beta y) \\ &= \alpha(\beta(x,y)).\end{aligned}$$

- I2: Let $(x, y) \in S$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}(\alpha+\beta)(x,y) &= ((\alpha+\beta)x, (\alpha+\beta)y) \\ &= (\alpha x + \beta x, \alpha y + \beta y) \qquad \text{by distribution of in } \mathbb{R} \\ &= (\alpha x, \alpha y) + (\beta x, \beta y) \\ &= \alpha(x,y) + \beta(x,y).\end{aligned}$$

2. Let V be a vector space and let $\mathbf{0}$ be the zero vector in V. Below is a proof that $\alpha\mathbf{0} = \mathbf{0}$ for any scalar α. Fill in the blanks with the axiom used at each step:

proof. Let α be a scalar and let $-(\alpha\mathbf{0})$ be the additive inverse of $\alpha\mathbf{0}$. Then

$$\begin{aligned}\mathbf{0} &= \alpha\mathbf{0} + [-(\alpha\mathbf{0})] && \underline{\text{A4: Additive inverse}} \\ &= \alpha(\mathbf{0}+\mathbf{0}) + [-(\alpha\mathbf{0})] && \underline{\text{A3: Zero vector}} \\ &= (\alpha\mathbf{0}+\alpha\mathbf{0}) + [-(\alpha\mathbf{0})] && \underline{\text{I1: Scalar distribution 1}} \\ &= \alpha\mathbf{0} + (\alpha\mathbf{0} + [-(\alpha\mathbf{0})]) && \underline{\text{A2: Associativity of addition}} \\ &= \alpha\mathbf{0} + \mathbf{0} && \underline{\text{A4: Additive inverse}} \\ &= \alpha\mathbf{0}. && \underline{\text{A4: Zero vector}}\end{aligned}$$

3. Let V be a vector space. For any $\mathbf{v} \in V$ let $-\mathbf{v}$ denote the additive inverse of $\mathbf{v}$. Prove that $-(-\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v} \in V$.
(Hint: consider $\mathbf{v} + [-\mathbf{v}] + [-(-\mathbf{v})]$ and simplify in two different ways).

Solution: Let $\mathbf{v} \in V$. Then

$$\begin{aligned}\Big(\mathbf{v} + [-\mathbf{v}]\Big) + [-(-\mathbf{v})] &= \mathbf{v} + \Big([-\mathbf{v}] + [-(-\mathbf{v})]\Big) && \text{(by axiom A2)} \\ &\downarrow \\ \mathbf{0} + [-(-\mathbf{v})] &= \mathbf{v} + \mathbf{0} && \text{(by axiom A4)} \\ &\downarrow \\ -(-\mathbf{v}) &= \mathbf{v} && \text{(by axiom A3).}\end{aligned}$$

4. Let $n \geq 0$ be a fixed integer, let V be the set of polynomials of degree less than or equal to n, and let $F = \mathbb{R}$ be the field of scalars. Assume that addition and scalar multiplication are defined in the expected way. Prove that V satisfies axioms A0, A3, and S0 (this means that V is a **subspace** of the vector space of *all* polynomials).

Solution: Remember that a polynomial $f(x)$ has the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some integer $n \geq 0$ and scalars $a_0, \ldots, a_n$. We say that $\boldsymbol{f(x)}$ **is of degree** $\boldsymbol{k}$ if $a_k \neq 0$ and $a_i = 0$ for all $i > n$ (so k is the largest power of x occurring in $f(x)$). A polynomial of degree less than or equal to k must have $a_i = 0$ for all $i > k$, but may also have $a_k = 0$.

Now we prove the three axioms:

- A0: Let $f, g \in V$. Then f and g can be written as

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ g(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \end{aligned}$$

  where $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are scalars. Then

$$\begin{aligned} f(x) + g(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \end{aligned}$$

  so $f + g$ is a polynomial of degree less than or equal to n. Therefore $f + g \in V$.
- A3: Let $z(x) = 0$. Then z is a polynomial of degree 0, which is less than or equal to n, so $z \in V$. It is easy to check that $f + z = f$ for any polynomial f.
- S0: Let $\beta \in \mathbb{R}$ be a scalar and define $f(x)$ as we did above. Then

$$\begin{aligned} \alpha f(x) &= \beta(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) \\ &= (\beta a_0) + (\beta a_1)x + (\beta a_2)x^2 + \cdots + (\beta a_n)x^n \end{aligned}$$

  so αf is a polynomial of degree less than or equal to n.

5. The set $V = \mathbb{C}$ of complex numbers can be thought of as a vector space with scalars in $\mathbb{R}$. Addition and scalar multiplication are defined by standard addition and multiplication in $\mathbb{C}$ (you should check that this is a vector space, but you do *not* need to write the proof here).

(a) Find two vectors $\mathbf{u}$ and $\mathbf{v}$ in V such that any element of V can be written as

$$\alpha\mathbf{u} + \beta\mathbf{v}$$

for some scalars $\alpha, \beta \in \mathbb{R}$.

Solution: The simplest solution is $\mathbf{u} = 1 \in \mathbb{C} = V$ and $\mathbf{v} = i \in \mathbb{C} = V$. We know that any complex number $\mathbf{z} \in \mathbb{C} = V$ can be written as the sum of a real part and an imaginary part:

$$\mathbf{z} = \alpha + \beta i.$$

for some $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} \mathbf{z} &= \alpha + \beta i \\ &= \alpha(1) + \beta(i) \\ &= \alpha\mathbf{u} + \beta\mathbf{v}. \end{aligned}$$

$\alpha\mathbf{u} + \beta\mathbf{v}$ is called a **linear combination of u and v.**

(b) Is it possible to find a single vector $\mathbf{w} \in V$ such that every vector of V can be written as $\alpha\mathbf{w}$ for some scalar α? If so, find such a $\mathbf{w}$. If not, explain why or give a counter example.

Solution: It is not possible to find such a $\mathbf{w}$. To see why, suppose that such a $\mathbf{w}$ exists. Then $1 \in \mathbb{C}$ can be written as a scalar multiple of $\mathbf{w}$:

$$\alpha\mathbf{w} = 1 \qquad \text{for some } \alpha \in \mathbb{R}.$$

Now multiply both sides by $\frac{1}{\alpha}$:

$$\mathbf{w} = \frac{1}{\alpha} \qquad \text{for some } \alpha \in \mathbb{R}.$$

This means that $\mathbf{w} = \frac{1}{\alpha}$ is a real number (since α is a real number). Since $\mathbf{w}$ is a real number, any scalar multiple of $\mathbf{w}$ is also a real number (since the field of scalars is $\mathbb{R}$). This means that $i \in \mathbb{C}$ cannot be a scalar multiple of $\mathbf{w}$. This is a contradiction, because we assumed that every complex number could be written as a scalar multiple of $\mathbf{w}$. Therefore no such $\mathbf{w}$ exists.