MTH 342 Worksheet 5

Week 4 – 10/24/2019

Name: Answer Key **Recitation time:** ________

1. The derivative operator $\frac{d}{dx}$ is a linear map from the space of all differentiable functions $\mathbb{R} \to \mathbb{R}$ to the space of all functions $\mathbb{R} \to \mathbb{R}$. What is the null space of $\frac{d}{dx}$? What is the nullity of $\frac{d}{dx}$?

Solution: The null space of $\frac{d}{dx}$ is the set of differentiable functions $f(x)$ such that $\frac{d}{dx}f(x) = 0$. The only such functions are the constant functions, so $\text{null}(\frac{d}{dx})$ is the set of constant functions.

Every constant function can be written as a scalar multiple of the function $g(x) = 1$, so $\text{null}(\frac{d}{dx})$ has basis $\{g(x)\}$. The nullity of $\frac{d}{dx}$ is

$$\dim_{\mathbb{R}}(\text{null}(\tfrac{d}{dx})) = 1$$

2. Let $V = \mathbb{R}^4$ and

$$W = \left\{a + bx + cx^2 : a, b, c \in \mathbb{R}, 2a + 2b - c = 0\right\}$$

viewed as real vector spaces.

a.) Construct a linear map $f : V \to W$ with rank 2.

Solution: There are many possible answers here.

Let $(a, b, c, d) \in \mathbb{R}^4$. The easiest way to ensure f has rank **less than or equal to** 2 is to only use 2 components (say a and b) in the output. For example, we could choose

$$f(a, b, c, d) = a + bx + (2a + 2b)x^2.$$

Notice that the $(2a + 2b)x^2$ term is included so that the image of f is contained in W. We will see by part (b) that $\dim(\text{range}(f)) = 2$, so $\text{rank}(f) = 2$.

b.) Find a basis for $\text{range}(f)$.

Solution: This answer depends on your answer to part (a). In our case,

$$\begin{aligned}
\text{range}(f) &= \{f(a, b, c, d) : a, b, c, d \in \mathbb{R}\} \\
&= \{a + bx + (2a + 2b)x^2 : a, b \in \mathbb{R}\} \\
&= \{a(1 + 2x^2) + b(x + 2x^2) : a, b \in \mathbb{R}\} \qquad (*).
\end{aligned}$$

We propose $B_1 = \{1 + x^2, x + x^2\}$ as a basis for $\text{range}(f)$. Notice that

$$\begin{aligned}
f(1, 0, 0, 0) &= 1 + 2x^2 \\
f(0, 1, 0, 0) &= x + 2x^2
\end{aligned}$$

so $B_1 \subset \text{range}(f)$. We can see from $(*)$ that $\text{range}(f) = \text{span}(B_1)$.

Finally, we need to check that B_1 is linearly independent. Let $c_1, c_2 \in \mathbb{R}$ and suppose

$$c_1(1 + 2x^2) + c_2(x + 2x^2) = 0.$$

Setting $x = 0$ gives $c_1 = 0$, so the above equation becomes

$$c_2(x + 2x^2) = 0.$$

Setting $x = 1$ gives $3c_2 = 0$, so $c_2 = 0$. Therefore the functions are linearly independent.

c.) What is the nullity of f?

Solution: This answer does **not** depend on your answer to part (a).

By the rank-nullity theorem

$$\begin{aligned}
\text{rank}(f) + \text{nullity}(f) &= \dim(V) \\
&\downarrow \\
2 + \text{nullity}(f) &= 4 \\
&\downarrow \\
\text{nullity}(f) &= 2
\end{aligned}$$

d.) Find a basis for null(f).

Solution: This answer depends on your answer to part (a).

We want to find all $(a, b, c, d) \in \mathbb{R}$ such that

$$\begin{aligned}
0 &= f(a, b, c, d) \\
&= a + bx + (2a + 2b)x^2 \\
&= a(1) + b(x) + (2a + 2b)x^2.
\end{aligned}$$

Since the set $\{1, x, x^2\}$ is linearly independent, the only way for this to be true is if $a = b = 0$. Therefore $(a, b, c, d) \in \text{null}(f)$ if and only if $a = b = 0$. That is,

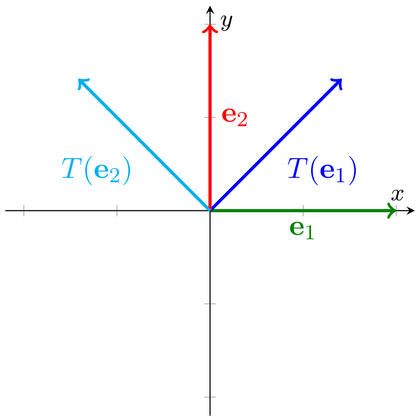
$$\text{null}(f) = \{(0, 0, c, d) : c, d \in \mathbb{R}\}.$$

It is easy to check that $B_2 = \{(0, 0, 1, 0), (0, 0, 0, 1)\}$ is a basis for null(f).

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map that rotates vectors by 45 degrees counter-clockwise about the origin.

a.) Find the standard matrix for T.

Solution: By standard matrix, we mean the matrix $[T]_{B_2,B_1}$ where B_1 and B_2 are both the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0),(0,1)\}$ for $\mathbb{R}^2$.



Using some basic trigonometry, we get

$$\begin{aligned} T(\mathbf{e}_1) &= (\cos(45^\circ), \sin(45^\circ)) \\ &= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{2}\mathbf{e}_1 + \frac{\sqrt{2}}{2}\mathbf{e}_2 \end{aligned}$$

so

$$[T(\mathbf{e}_1)]_{B_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

A similar calculation gives

$$\begin{aligned} T(\mathbf{e}_1) &= (\cos(135^\circ), \sin(135^\circ)) \\ &= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \\ &= -\frac{\sqrt{2}}{2}\mathbf{e}_1 + \frac{\sqrt{2}}{2}\mathbf{e}_2 \end{aligned}$$

so

$$[T(\mathbf{e}_2)]_{B_2} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Then the standard matrix for T is

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

b.) Use the standard matrix to find $T(3,1)$.

Solution: If A is the standard matrix of T and $\mathbf{v} \in \mathbb{R}^2$ then $A\mathbf{v}$ gives the standard coordinates of $T(\mathbf{v})$. In our case we calculate

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\ 3\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$$

so

$$\begin{aligned} T(3,1) &= (\sqrt{2})\mathbf{e}_1 + (2\sqrt{2})\mathbf{e}_2 \\ &= (\sqrt{2}, 2\sqrt{2}) \end{aligned}$$

c.) What is the nullity of T? What is the rank of T?

Solution: The only element that maps to $(0,0)$ under T is $(0,0)$, so

$$\text{null}(T) = \{(0,0)\}$$

and $\text{nullity}(f) = 0$. The rank-nullity theorem gives

$$\text{rank}(T) = \dim(\mathbb{R}^2) - \text{nullity}(T) = 2 - 0 = 2.$$

4. Does there exist a linear map $f\colon \mathbb{R}^4 \to \mathbb{R}$ with null space

$$N = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 3x_2,\ x_3 = x_4\}?$$

If yes, give an example of such a linear map. If no, explain why.

Solution: No such linear map exists. Notice that we can rewrite N as

$$\begin{aligned} N &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 3x_2,\ x_3 = x_4\} \\ &= \{(3x_2, x_2, x_3, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{x_2(3,1,0,0) + x_3(0,0,1,1) : x_2, x_3 \in \mathbb{R}\} \end{aligned}$$

It is not difficult to show that this set has basis $B = \{(3,1,0,0), (0,0,1,1)\}$, so $\dim(N) = 2$.

Suppose N is the null space of $f\colon \mathbb{R}^4 \to \mathbb{R}$. Then

$$\text{nullity}(f) = \dim(N) = 2.$$

By the rank nullity theorem we must have

$$\begin{gathered} \text{rank}(f) + \text{nullity}(f) = \dim(\mathbb{R}^4) \\ \downarrow \\ \text{rank}(f) + 2 = 4 \\ \downarrow \\ \text{rank}(f) = 2 \end{gathered}$$

But this is impossible, because f maps to a 1-dimensional space, so $\text{rank}(f)$ is less than or equal to 1. Therefore our assumption that N is the null space of f was incorrect.