MTH 342 Worksheet 6

Week 5 – 10/31/2019

Name: Answer Key **Recitation time:** __________

1. Let V, W be vector spaces over a field F and let $f\colon V \to W$ and $g\colon W \to V$ be linear maps such that

$$g \circ f(\mathbf{v}) = \mathbf{v} \qquad \text{for all } \mathbf{v} \in V$$

(a) Prove that f is a monomorphism.

Solution: Let 0_V and 0_W represent the zero vectors in V and W respectively. To prove f is a monomorphism, we want to show that $\text{null}(f) = \{0_V\}$.

Let u be an arbitrary element of (f). Then $f(u) = 0_W$, so

$$\begin{aligned} u &= g \circ f(u) \\ &= g(f(u)) \\ &= g(0_W) \\ &= 0_V. \end{aligned}$$

Therefore $\text{null}(f) = \{0_V\}$.

(b) Prove that g is an epimorphism.

Solution: Let v be an arbitrary element of V. To prove that g is an epimorphism, we must show that there exists some $w \in W$ such that $g(w) = v$.

Let $w = f(v)$. Then

$$\begin{aligned} g(w) = &= g(f(v)) \\ &= g \circ f(v) \\ &= v \end{aligned}$$

(c) What can we conclude about the relationship between $\dim(V)$ and $\dim(W)$?

Solution: We can conclude that $\dim(W) \geq \dim(V)$.

In order to reach a contradiction, assume instead that $\dim(W) < \dim(V)$. The rank-nullity theorem tells us

$$\begin{aligned} \text{rank}(f) + \text{nullity}(f) &= \dim(V) \\ &\downarrow \\ \text{nullity}(f) &= \dim(V) - \text{rank}(f) \\ &\geq \dim(V) - \dim(W) && \text{since } \text{rank}(f) \leq \dim(W) \\ &> 0 && \text{since } \dim(W) < \dim(V) \end{aligned}$$

Therefore f is not injective (i.e., not a monomorphism), contradicting part (a).

A similar proof can be constructed using the rank-nullity theorem applied to g.

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map that reflects each vector across the line $y = -x$ and let $S\colon P_2(\mathbb{R}) \to \mathbb{R}^2$ be the map defined by

$$S(u) = (u(0), u(2)).$$

(a) Find matrix representations for T, S, and TS.

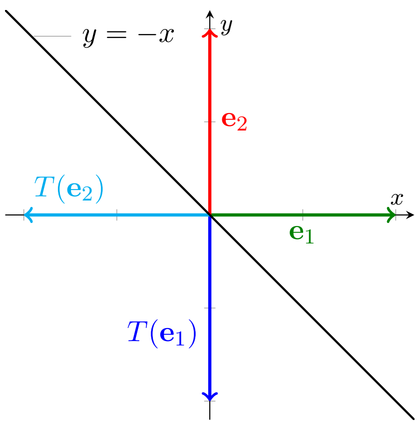
Solution: To find the matrix representations, we must choose a basis for $\mathbb{R}^2$:

$$\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$$

and a basis for $P_2(\mathbb{R})$:

$$\mathcal{B}_2 = \{x^2, x, 1\}$$

We can now find the matrix representations with respect to these bases.



$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\mathbf{e}_2 = 0\mathbf{e}_1 + (-1)\mathbf{e}_2,$$
$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\mathbf{e}_1 = (-1)\mathbf{e}_1 + 0\mathbf{e}_2$$

The matrix for T is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

$$\begin{aligned} S(x^2) &= (0^2, 2^2) \\ &= (0, 4) \\ &= 0\mathbf{e}_1 + 4\mathbf{e}_2 \end{aligned} \qquad \begin{aligned} S(x) &= (0, 2) \\ &= 0\mathbf{e}_1 + 2\mathbf{e}_2 \end{aligned} \qquad \begin{aligned} S(1) &= (1, 1) \\ &= 1\mathbf{e}_1 + 1\mathbf{e}_2 \end{aligned}$$

The matrix for S is

$$\begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}.$$

The matrix for TS is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) Which of the maps from part (a) are epimorphisms? Monomorphisms? Isomorphisms?

Solution:

- It is easy to see that T is its own inverse. Therefore by problem 1 parts (a) and (b), T is both a monomorphism and an epimorphism. Therefore T is an isomorphism.
- The map S is not a monomorphism, because

  $$\dim(P_2(\mathbb{R})) = 3 > 2 = \dim(\mathbb{R}^2).$$

  However, S is an epimorphsim, because the set

  $$\{(1, 1), (0, 2)\} = \{S(1), S(x)\} \subset \operatorname{range}(S)$$

  spans all of $\mathbb{R}^2$ (so $\operatorname{range}(S) = \mathbb{R}^2$).
- TS is also a monomorphism but not an epimorphism. The arguments are similar to those given for the map S.