Worksheet

11/06/2019

1. Let

$$
\begin{aligned}
U &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 = 0\}, \\
V &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 = x_2 + x_3\}.
\end{aligned}
$$

   Show that $U + V = \mathbb{R}^3$.

See lecture notes for solution to this problem.

2. Let

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = 0, \ x_2 = x_3\}, \\ V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = x_3 + x_4, \ x_2 = x_3\}. \end{aligned}$$

Find a basis of $U + V$.

Step 1: find a basis of U.

$$U = \{(0, x_2, x_2, x_4) : x_2, x_4 \in \mathbb{R}\} = \{x_2(0,1,1,0) + x_4(0,0,0,1) : x_2, x_4 \in \mathbb{R}\}$$

$$= \text{span}\{\underbrace{(0,1,1,0)}_{u_1}, \underbrace{(0,0,0,1)}_{u_2}\}.$$

One can check that u_1 and u_2 are linearly independent.

Thus, $\{u_1, u_2\}$ is a basis of U.

Step 2: find a basis of V.

$$V = \{(x_3 + x_4, x_3, x_3, x_4) : x_3, x_4 \in \mathbb{R}\}$$

$$= \cdots$$

$$= \text{span}\{\underbrace{(1,1,1,0)}_{v_1}, \underbrace{(1,0,0,1)}_{v_2}\}$$

Step 3: $U + V = \text{Col}(A)$ where

$$A = \begin{bmatrix} | & | & | & | \\ u_1 & u_2 & v_1 & v_2 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

After reducing A to RREF, we get

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow \end{matrix}$$

pivot columns

The 1st, 2nd, 3rd columns of A form a basis of $\text{Col}(A)$. Therefore, a basis of $U + V$ is $\{u_1, u_2, v_1\}$.