Worksheet

11/18/2019

1. Let P_3 be the vector space of all polynomials of degree ≤ 3 with real coefficients. Let $F : P_3 \to P_3$ be a linear map given by $F(u) = xu' + u$. Consider the following subspaces:

$$\begin{aligned} V_1 &= \{u \in P_3 : \ u(0) = 0\}, \\ V_2 &= \{u \in P_3 : \ u(1) = 0\}. \end{aligned}$$

Which of them is invariant under F?

We will attempt to show that V_1 is invariant under F. If we can't do that, we will switch to showing that V_1 is not invariant under F.

Take $u \in V_1$. We want to show $F(u) \in V_1$. That is to show $F(u)(0) = 0$. That is to show

$$\left.(xu' + u)\right|_{x=0} = 0. \qquad (*)$$

We see that

$$LHS(*) = 0\,u'(0) + u(0) = u(0)$$

which is equal to 0 because $u \in V_1$. Thus, $(*)$ is true. We conclude that V_1 is indeed invariant under F.

Now we also attempt to show that V_2 is invariant under F. Take $u \in V_2$. We want to show $F(u)(1) = 0$. That is to show

$$\left.(xu' + u)\right|_{x=1} = 0 \qquad (**)$$

We have

$$LHS = 1\,u'(1) + \underbrace{u(1)}_{=0 \text{ due to } u \in V_2} = u'(1).$$

We see that LHS $= u'(1)$, not zero. For this reason we now try to select a specific $u \in V_2$ such that $F(u) \notin V_2$. Once this is done, we conclude that V_2 is not invariant under F.

We need to find $\underbrace{u \in V_2}_{\substack{\text{degree} \leq 3, \\ \text{vanishing at } 1}}$ such that $u'(1) \neq 0$.

Let us pick $u = x - 1$.

2. Consider the following subspaces of $M_{2\times 2}(\mathbb{R})$

$$V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a + d = b + c = 0 \right\}, \qquad V_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a + c = b = d = 0 \right\},$$

$$V_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ b = c = d = 0 \right\}.$$

Show that $V_1 \oplus V_2 \oplus V_3 = M_{2\times 2}(\mathbb{R})$.

See solution to this problem in lecture notes.