Problem 1.

Let (x_n) be a sequence defined recursively by $x_{n+1} = \frac{1}{4}\left(x_n^3 - 3x_n + 6\right)$.

a) Suppose that x_n has a limit $a \in \mathbb{R}$. Find all possible values of a.

b) What is the function g such that $x_{n+1} = g(x_n)$? Graph this function using Matlab on the interval $(-4, 3)$.

c) Let $x_0 = 1.5$. Draw roughly (by hand) a cobweb diagram of the sequence (x_n).

d) Which of the fixed points are stable? Which are unstable?

e) Given that the sequence $x_n \to 1$ as $n \to \infty$, find the order of convergence of the sequence (x_n).

Solution

a) The limit of the sequence is often dependent on the initial term of the sequence, x_0. We could check every possible $x_0 \in \mathbb{R}$, however this could prove quite tedious. By taking the limit of both sides of the equation as $n \to \infty$, we get an equation to solve for a:

$$a = \frac{1}{4}(a^3 - 3a + 6) \implies 0 = a^3 + -3a - 4a + 6 \implies 0 = \frac{1}{4}(a-2)(a-1)(a+3)$$

Which has three roots, $a = -3, 1, 2$. Such a point is said to be an equilibrium state of the iteration scheme because if $x_0 = a$, then $x_1 = x_2 = \ldots = a$.

b) We can collect the right side of the iteration scheme into a function, $g(x) = \frac{1}{4}(x^3 - 3x + 6)$.

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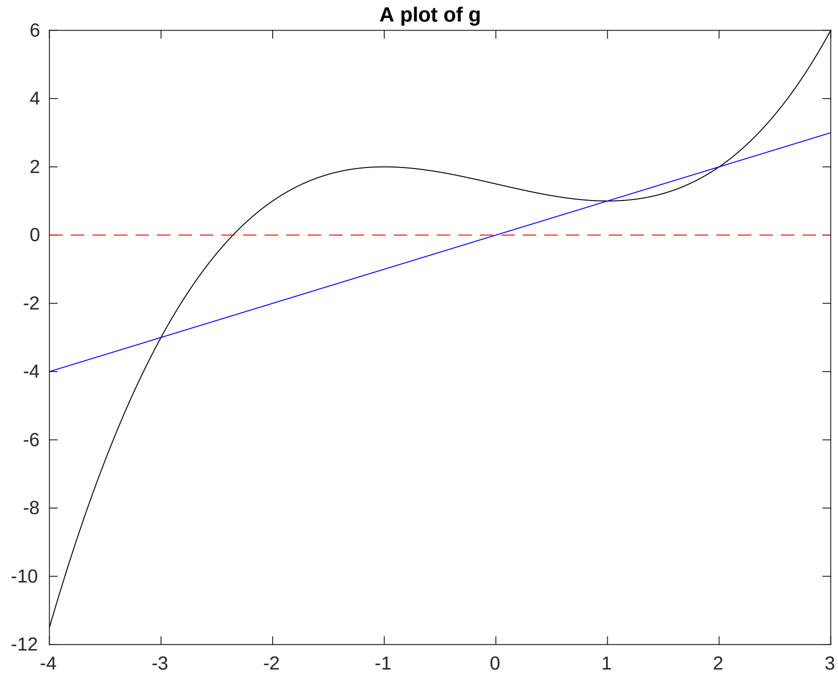
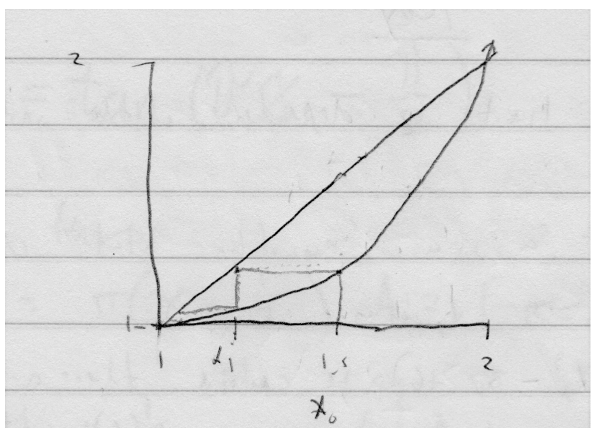
%MTH 351 HW5 Problem 1b
x = linspace(-4,3,700);
zero = zeros(1, length(x));
y = 1/4*(x.^3-3*x+6);
plot(x,y,'k')
hold on
plot(x,zero,'r--')  % Add a horizontal axis
plot(x,x,'b')  % Add the identity line to the map
title('A plot of g')
hold off
```

You do not need to include the blue or red lines to obtain full credit, however they may be useful to refer to later on.



c)



The cobweb diagram should start at $x_0 = 1.5$, and converge to $x = 1$. $g(x) < x$ for $x \in (1, 2)$ characterizes the plot of g.

d) We can apply the Contraction mapping theorem (3.4.2, page 98) to determine stability of our candidate fixed points, $a = -3, 1, 2$. By the cobweb diagram, we can guess that $a = 1$ is stable, while $a = 2$ and $a = 3$ are unstable. One can verify these observations rigorously as follows.

First we need to find some interval $[a, b]$ that contains 1 (our fixed point) and satisfies $\max\limits_{a \le x \le b} |g'(x)| < 1$. The interval $[0.5, 1.5]$ will do nicely. $0.5 < g(0.5) \approx 1.1562 < 1.5$ and $0.5 < g(1.5) = 1.2188 < 1.5$. Next,

$$g'(x) = \frac{3}{4}(x^2 - 1) \implies \max_{a \le x \le b} |g'(x)| = g'(1.5) = 0.9375 < 1$$

and g and g' are polynomials, so they are continuous functions. We have an interval that satisfies all requirements of the contraction mapping theorem. We can conclude by this theorem that there is a unique fixed point in $[0.5, 1.5]$ and this fixed point is stable for any initial $x_0 \in [a, b]$. For a point to be stable, we only need to find some small interval around it (often referred to as a neighborhood) such that any x_0 in that interval converges to 1. So $a = 1$ is stable.

The other two cases are not nearly as generous. Let x_0 be close to 2. Then

$$|x_{n+1} - 2| = \frac{1}{4}|x_n^3 - 3x_n + 6 - 8| = \frac{1}{4}|x_n^3 - 3x_n - 2| = \frac{1}{4}|(x_n - 2)^2(x_n + 1)^2|$$

If we assume that x_n is close to 2, then $x_n + 1 \approx 3$, so

$$|x_{n+1} - 2| \approx \frac{1}{4}|(x_n - 2)|(3)^2 = \frac{9}{4}|x_n - 2|$$

and so we arrive at what would appear to be a convergence statement, however $C = 9/4 = 2.25 > 1$, so the fixed point $a = 2$ is unstable.

We can use a similar method to show that $a = -3$ is also an unstable fixed point.

$$|x_{n+1} + 3| = \frac{1}{4}|x_n^3 - 3x_n + 6 + 12| = \frac{1}{4}|x_n^3 - 3x_n + 18| = \frac{1}{4}|(x + 3)(x^3 - 3x + 6)|$$

If we assume $x \approx -3$, then $x^2 - 3x + 6 \approx 9 + 9 + 6 = 24$, so

$$|x_{n+1} + 3| \approx \frac{24}{4}|x_n + 3| = 6|x_n + 3|$$

Which appears to be a convergence statement, however as $C = 6 > 1$, the fixed point $a = -3$ is unstable.

So $a = 1$ is the stable fixed point and $a = -3, 2$ are unstable fixed points.

e)

$$|x_{n+1} - 1| = \frac{1}{4}|x_n^3 - 3x_n + 6 - 4| = \frac{1}{4}|x_n^3 - 3x_n + 2| = \frac{1}{4}|(x_n - 1)^2(x_n + 2)|$$

If we assume $x_n \approx 2$ (but $x_n \neq 2$), then $x_n + 2 \approx 3$, so

$$|x_{n+1} - 1| \lessapprox \frac{3}{4}|(x_n - 1)^2| = \frac{3}{4}|x_n - 1|^2$$

And we find that the order of convergence of the sequence of x_n is 2.

Problem 2.

In this problem, we want to find a polynomial curve passing through the four points $(1, 0), (2, 2), (3, 0), (4, 1)$.

a) Find a polynomial P whose graph passes through the given points. Make sure to simplify P.

b) Use Matlab to plot the graph of P on the interval $(0, 5)$.

c) Find $P(1.5)$ and $P'(1.5)$.

Solution

a) Using Lagrange's interpolation method may save us some calculations, as

$$P(x) = 0L_1(x) + 2L_2(x) + 0L_3(x) + 1L_4(x) = 2L_2(x) + L_4(x)$$

so we only need to compute two basis polynomials.

$$L_2(x) = \frac{(x-1)(x-3)(x-4)}{1 \cdot -1 \cdot -2} = \frac{1}{2}(x-1)(x-3)(x-4)$$

and

$$L_4(x) = \frac{(x-1)(x-2)(x-3)}{3 \cdot 2 \cdot 1} = \frac{1}{6}(x-1)(x-2)(x-3)$$

so

$$P(x) = \frac{2}{2}(x-1)(x-3)(x-4) + \frac{1}{6}(x-1)(x-2)(x-3) = \frac{7}{6}x^3 - 9x^2 + \frac{125}{6}x - 13$$

b) Full credit will require the plot to be submitted on paper.

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%MTH 351 HW5 Problem 1b
x = linspace(0,5,700);
zero = zeros(1, length(x));
y = 7/6*x.^3 - 9*x.^2 + 125/6*x - 13;
plot(x, y, 'k')
hold on
grid on
% Add in the points we want to interpolate as a sanity check
scatter(1, 0, 'g', 'filled')
scatter(2, 2, 'b', 'filled')
scatter(3, 0, 'r', 'filled')
scatter(4, 1, 'm', 'filled')
title('A plot of P')
hold off
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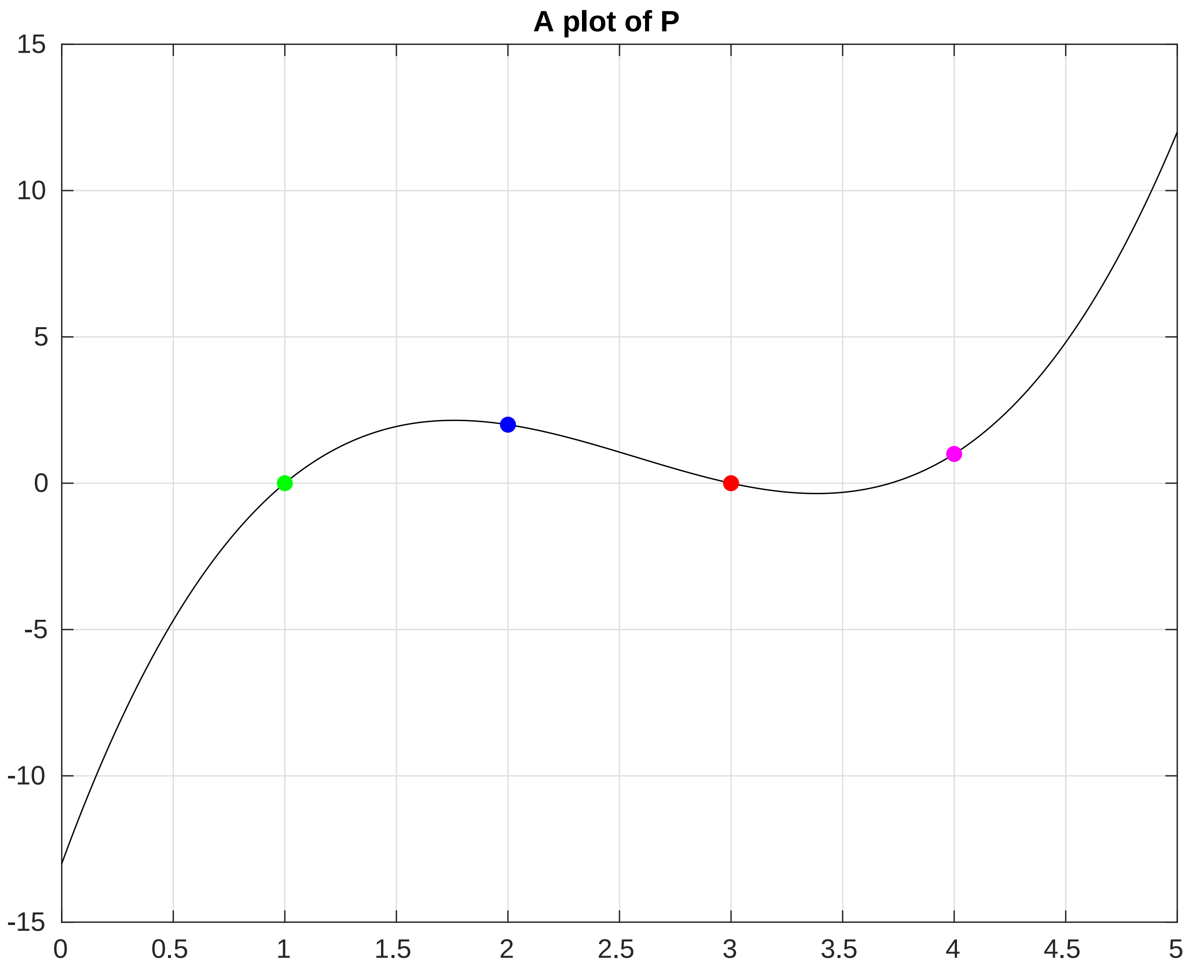

c)

$$P(1.5) = \frac{31}{16} = 1.9375, \quad P'(1.5) = \frac{41}{24} = 1.7083\bar{3}$$