Homework 7

Due 11/25/2019

1. Given a function f on some interval, say $[-1,1]$, and an integer $N > 1$, we are interested in the question: *what set of sample points* $\{x_1, x_2, \ldots, x_n\}$ *on* $[-1,1]$ *should we choose so that the interpolation polynomial* P_n *can best approximate function* f*?* Note that the number of sample points n is fixed. We are free to choose the sample points.

   To investigate this question, let us consider an example $f(x) = \frac{1}{1+10x^2}$ and $n = 11$. Consider two different ways of sampling:

   - Evenly spaced $-1 = x_1 < x_2 < \ldots < x_n = 1$,
   - Unevenly spaced $z_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for $k = 1, 2, \ldots, n$.

   (a) Use the command **Plot** to sketch each set of sample points on the interval $[-1,1]$.

   (b) Let P_n be the interpolation polynomial corresponding to the set of data points $(x_1, f(x_1))$, $\ldots$, $(x_n, f(x_n))$. Plot P_n and f on the same graph.

   (c) Let Q_n be the interpolation polynomial corresponding to the set of data points $(z_1, f(z_1))$, $\ldots$,$(z_n, f(z_n))$. Plot Q_n and f on the same graph.

   (d) Based on the graphs, is one way of sampling significantly better than the other? Give a rough explanation for your observation.

   (e) The same questions in Parts (b), (c), (d) but for $f(x) = \cos x$.

2. Interpolation gives an alternative method to approximate a function f by polynomials (other than Taylor approximation method). In this exercise, we investigate error estimates of this method. Let
$$f(x) = e^{\frac{x}{2}} \sin\left(\frac{x}{2}\right).$$
   For evenly spaced sample points $0 = x_1 < x_2 < \ldots < x_n = 4$, let P_n be the corresponding interpolation polynomial.

   (a) Show that $|f'(x)| \le e^{x/2}$ and $|f''(x)| \le e^{x/2}$.

   (b) It is known that (you don't have to verify) $|f^{(k)}(x)| \le e^{x/2}$ for any $x \in \mathbb{R}$ and $k \ge 1$. Find n such that
$$|f(x) - P_n(x)| \le 10^{-4} \quad \forall\, x \in [0,4].$$

   (c) Find n such that the integral $\int_0^4 P_n(x)dx$ approximates $\int_0^4 f(x)dx$ with error not exceeding 10^{-3}.
   Hint: use the inequality
$$\left|\int_a^b (f(x) - g(x))dx\right| \le \int_a^b |f(x) - g(x)|dx.$$

3. Let $f(x) = \frac{1}{x+1}$. For evenly spaced sample points $0 = x_1 < x_2 < \ldots < x_n = 2$, let P_n be the corresponding interpolation polynomial. Find n such that
$$|f(x) - P_n(x)| \le 10^{-4} \quad \forall\, x \in [0,2].$$