Denote $I = \int_0^2 \frac{1}{4+x^2}\,\mathrm{d}x$.

Problem 1.

Find the exact value of I.

Solution

$$\int_0^2 \frac{1}{4+x^2}\,\mathrm{d}x = \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^2 = \frac{1}{2}\arctan(1) - \arctan(0) = \frac{\pi}{8}$$

Problem 2.

For a generic positive integer n we take $n+1$ equally spaced sample points indexed by $x_0, x_1, \ldots, x_n$ on the interval $[0,2]$. Denote by L_n, R_n, M_n, T_n the Riemann sums corresponding to the left-point, right-point, midpoint, and trapezoid rule. Use sigma notation to write a formula for each L_n, R_n, M_n, T_n.

Solution

$$L_n = \sum_{i=0}^{n-1} \frac{2}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{2}{n}\frac{1}{4+x_i^2} = \sum_{i=0}^{n-1} \frac{2}{n}\frac{1}{4+\left(i\frac{2}{n}\right)^2} = \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{n+\frac{i^2}{n}}$$

$$R_n = \sum_{i=1}^{n} \frac{2}{n} f(x_i) = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{n+\frac{i^2}{n}}$$

Note that the indexing has changed between L_n and R_n.

$$M_n = \sum_{i=0}^{n-1} \frac{2}{n}\frac{1}{4+\left(\frac{2i+1}{n}\right)^2} = \sum_{i=0}^{n-1} \frac{2}{1}\frac{1}{4n+\frac{1}{n}(2i+1)^2} = \sum_{i=0}^{n-1}\frac{n}{4n^2+(2i+1)^2}$$

$$T_n = \sum_{i=0}^{n-1} \frac{2}{n}\frac{f(x_i)+f(x_{i+1})}{2} = \sum_{i=0}^{n-1}\frac{1}{n}\left(\frac{1}{4+\left(\frac{2i}{n}\right)^2} + \frac{1}{4+\left(\frac{2i+2}{n}\right)^2}\right) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{n^2\left(2i(i+1)+2n^2+1\right)}{4\left(i^2+n^2\right)\left((i+1)^2+n^2\right)}$$

(Please simplify your answer so the sum is entirely in terms of an index variable (i above) and n.)

Problem 3.

Which of these three methods gives the best approximation of I when $n=4$?

Solution

We can use the above formulas to compute the approximations, then find the error bounds.

$$|L_4 - \pi/8| \approx 0.029947877124805$$
$$|R_4 - \pi/8| \approx 0.032552022875195$$
$$|M_4 - \pi/8| \approx 6.509831005186983 \times 10^{-4}$$
$$|T_4 - \pi/8| \approx 0.001302022875195$$

M_4 gives the best approximation of the four choices.

Problem 4.

Write matlab code to compute L_n, R_n, M_n, and T_n for $n = 8, 16, 32, 64$.

Solution

We do this in Matlab. For simplicity, we write one script for each method which allows us to easily change n (line 3).

Left point method:

```
1 a = 0;
2 b = 2;
3 n = 4;
4 xvals = linspace(a,b,n+1);  % Generate n+1 points
5 yvals = objective(xvals);
6 total = 0;
7 for ii = 1:n
8     total = total + (yvals(ii))*(b-a)/n;
9 end
10 disp(total)
11 error = total - pi/8;
12 disp( error)
13 function out = objective(in)
14    out = 1./(4 + in.^2);
15 end
```

Right point method:

```
1 a = 0;
2 b = 2;
3 n = 4;
4 xvals = linspace(a,b,n+1);  % Generate n+1 points
5 yvals = objective(xvals);
6 total = 0;
7 for ii = 1:n
8     total = total + (yvals(ii+1))*(b-a)/n;
9 end
10 disp(total)
11 error = total - pi/8;
12 disp( error)
13 function out = objective(in)
14    out = 1./(4 + in.^2);
15 end
```

Trapezoidal method:

```
1 a = 0;
2 b = 2;
3 n = 4;
4 xvals = linspace(a,b,n+1);  % Generate n+1 points
5 yvals = objective(xvals);
6 total = 0;
7 for ii = 1:n
8     total = total + (yvals(ii) + yvals(ii+1))/2*(b-a)/n;
9 end
10 disp(total)
```

```
error = total - pi/8;
disp( error)
function out = objective(in)
   out = 1./(4 + in.^2);
end
```

Midpoint method:

```
a = 0;
b = 2;
n = 4;
xvals = linspace(a,b,n+1);  % Generate n+1 points
xvals = xvals + 1/n;  % Shift xvalues by one half
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii))*(b-a)/n;
end
disp(total)
error = total - pi/8;
disp( error)
function out = objective(in)
   out = 1./(4 + in.^2);
end
```

Problem 5.

Find values of n such that the error for each left point, right point, midpoint, and trapezoidal rule approximations are bounded by $\epsilon = 0.0001$.

Solution

We need to find bounds on K and $\widetilde{K}$.

$$f'(x) = \frac{-2x}{(4+x^2)^2} \implies |f'(x)| = \frac{2x}{(4+x^2)^2}, \quad x \in [0,2]$$

$$f''(x) = \frac{8x-6x^3}{(x)(4+x^2)^3}, \; f''(x) = 0 \implies 8x = 6x^3 \implies x = \pm\frac{2}{\sqrt{3}}$$

The point $x = \frac{2}{\sqrt{3}}$ is inside the desired interval, and maximizes the absolute value of $f'(x)$ (as $f'(x) > f'(0)$ for $x > 0$). Then we can bound $|f'(x)|$ on the interval by $K = \frac{3\sqrt{3}}{64} \approx 0.0811898816$, or choose any value larger. Likewise,

$$f'''(x) = \frac{24x(x^2-4)}{(4+x^2)^4}, \; f'''(x) = 0 \implies x(x^2-4) = 0 \implies x \in \{-2, 0, 2\}$$

And we check the endpoints to find that $f''(0) > f''(2)$ (-2 is not in the interval). So $|f''(x)| \leq \frac{1}{8} = \widetilde{K}$. You can choose any bound greater than this that you can justify. One simple method is as follows:

$$|f'(x)| = \frac{2x}{(4+x^2)^2} \leq \frac{2(2)}{(4+0^2)^2} = \frac{1}{4}.$$

$$|f''(x)| = \frac{|8-6x^2|}{(4+x^2)^3} \leq \frac{8+6x^2}{(4+x^2)^3} \leq \frac{8+6(2)^2}{(4+0^2)^3} = \frac{1}{2}.$$

Left point and right point methods have the same error bound.

$$e_n^{(L)}, e_n^{(R)} \leq \frac{K(2)^2}{2n} \leq \frac{(1/4)(2)^2}{2n} = \frac{1}{2n} \leq 0.0001 \Rightarrow n \geq 5000.$$

$$e_n^{(M)} \leq \frac{\tilde{K}(2)^3}{24n^2} \leq \frac{(1/2)(2)^3}{24n^2} = \frac{1}{6n^2} \leq 0.0001 \Rightarrow n \geq 41.$$

So choosing $n \geq 41$ is sufficient. Lastly,

$$e_n^{(T)} \leq \frac{\dot{\tilde{K}(2)^3}}{12n^2} = \frac{(1/2)(2)^3}{12n^2} = \frac{1}{3n^2} \leq 0.0001 \Rightarrow n \geq 58.$$

So choosing $n \geq 58$ is sufficient.