Lecture 1 (9/25/2019)

Many problems in real life can be formulated as solving a mathematical equation $f(x) = 0$.

f .... known function, procedure

x .... the unknown

Examples are numerous in sciences such as physics, biology, etc. Usually the function f is complicated. Let us consider a few examples where f is simple.

(1) $f(x) = ax + b \dashrightarrow x = -b/a$

(2) $f(x) = ax^2 + bx + c \dashrightarrow x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

(3) $f(x) =$ cubic function $\dashrightarrow$ an explicit root found by Cardano (1545)

(4) $f(x) =$ quartic function $\dashrightarrow$ Ferrari ($\sim$ 1545)

(5) $f(x) =$ poly. of degree ≥ 5 $\dashrightarrow$ cannot be solved by radicals (Ruffini, Abel 1820s)

Finding roots of polynomial has application in the problem of finding eigenvalues of a matrix, or solving linear ODE (ordinary differential equations). Trying to find roots inspired the invention of new systems of numbers. For example, (1) leads to the notion of rational numbers because the root $x = -b/a$ is not necessarily an integer even if a and b are integers. (2) leads to the notion of real numbers. (3) leads to complex numbers.

The formula of Cardano and Ferrari are often hard to use to find a numerical value of the roots. Also, they gave formula for only one root. It will take a lot more effort to find other roots by facturization. On the other hand, in practice one usually needs an approximate value of the root.

⇝ There is a need for more efficient methods to solve these equations.

Consider the following equation : $\underbrace{x^5 - 3x + 1}_{f(x)} = 0$

We see that $f(0) = 1$ and $f(1) = -1$. We know that f must have a root somewhere between 0 and 1. This root is probably close to the average of 0 and 1, which is 1/2. Let us define a sequence of approximate roots as follows :

$$x_0 = 1/2$$
$$x_1 = \frac{1}{3}(x_0^5 + 1)$$
$$x_2 = \frac{1}{3}(x_1^5 + 1)$$
$$\dots$$
$$x_{n+1} = \frac{1}{3}(x_n^5 + 1) \qquad (*)$$

In this iterative procedure, we can compute every x_n. Indeed, knowing x_0, we will know x_1. Knowing x_1, we will know x_2, and so on.

What happens if the sequence (x_n) has limit a? By taking the limit of both sides of $(*)$, one gets $a = \frac{1}{3}(a^5 + 1)$.

So a is a root that we were looking for!

* On Matlab :

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x = 1/2;
for n=1:10
    x = 1/3 (x^5+1)
end
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⇝ We observe that the sequence converges quite quickly, indicating that the iterative procedure is quite efficient.

* Numerical analysis:

  1) Approximation methods.

  2) Error analysis $\longrightarrow$ from the approximation methods
     $\searrow$ from computer (due to roundoff)

- Errors coming from computer are hard to control and relatively random in nature. This type of error is usually small. However, it may be magnified / accumulated if the procedure uses too many operations. There are also issues with floating-point arithmetic that we need to be aware of, for example, when we multiply a very small number by a very large number. Say, $\underbrace{x}_{\text{order } 10^{10}} \underbrace{(\sqrt{x+1} - \sqrt{x})}_{\text{order } 10^{-5}}$ when x is very large.

If the computer can store only 4 digits after the decimal point then $\sqrt{x+1} - \sqrt{x}$ will be rounded as 0. Thus the product gives 0, which is incorrect. However, if one writes

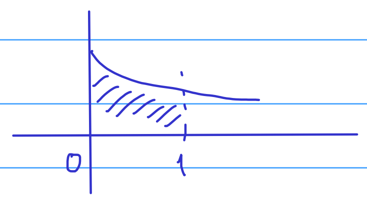
$$\frac{x}{\sqrt{x+1} + \sqrt{x}} \quad \text{(which is an equivalent form)}$$

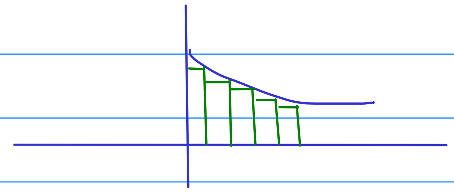
then there is no issue.

- What we can control, at least most of the time, are errors from appr. methods. This type of error usually dominates the errors from computer.

  Ex: Compute $\displaystyle\int_0^1 \cos(x^2)\,dx$

  The integrand doesn't have an antiderivation in closed form. Thus, one can't benefit from the Fundamental theorem of calculus. However, one can evaluate approximately. Such idea was employed by Archimedes in 300s BC, when he approximated the area of a circle by that of polygons. Riemann used a similar idea to approximate area under a general curve (1860s).





$$\int_0^1 \cos(x^2)\,dx \approx \text{Riemann sum}$$

Another way to appr. the integral is to use Taylor series.

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

$$\rightsquigarrow \cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

Then

$$\int_0^1 \cos(x^2)\,dx \approx \int_0^1 \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}\right) dx$$

which one can compute easily.

We will discuss how to control the error in this estimate next time.