Lecture 10 (10/16/2019)

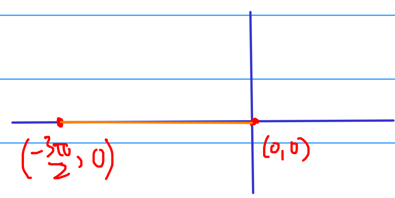
The bisection method can be flexibly used to find root of a continuous multivariable function.

Ex: find a root of the equation $e^{x_1+x_2} = \sin x_1 + \sin x_2$.
Consider the function

$$f(x_1, x_2) = e^{x_1+x_2} - \sin x_1 - \sin x_2.$$

This function is continuous on $\mathbb{R}^2$.
Let us pick two points where f has different signs:

$$f(0,0) = e^0 - \sin 0 - \sin 0 = 1 > 0$$
$$f\left(-\frac{3\pi}{2}, 0\right) = e^{-3\pi/2} - \sin\left(\frac{-3\pi}{2}\right) = e^{-3\pi/2} - 1 < 0$$



We then restrict f on the line segment connecting points $A(0,0)$ and $B\left(-\frac{3\pi}{2}, 0\right)$.

This line segment can be parametrized by one variable t as

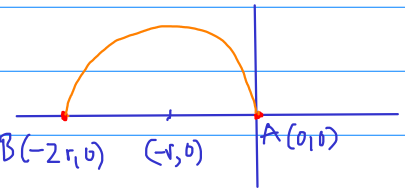
$$(x, y) = (t, 0)$$

Put $g(t) = f(t, 0) = e^{-t-0} - \sin t - \sin 0 = e^{-t} - \sin t$.
Then

$$g(0) = f(0,0) > 0, \quad g\left(-\frac{3\pi}{2}\right) = f\left(-\frac{3\pi}{2}, 0\right) < 0.$$

Then we can find a root of g between $-3\pi/2$ and 0 by the bisection method. This root of g corresponds to a root of f on the line segment in the picture.

One can also find a root of f on quite an arbitrarily arc connecting A and B. For example, let us look for a root of f on the semicircle above the segment AB.



$r = \frac{3\pi}{4}$

First, we need to parametrize this arc by <u>one</u> variable. There are many ways to do it. One way is the following:

$$\begin{cases} x = -r\cos t - r \\ y = -r\sin t - 0 = -r\sin t \end{cases}$$

Point A corresponds to $t = \pi$, B corresponding to $t = 0$. Then restrict f on this arc by putting

$$h(t) = f(x(t), y(t))$$

$$= e^{-r\cos t - r - r\sin t} - \sin(-r\cos t - r) - \sin(-r\sin t)$$

$$= e^{-\frac{3\pi}{4}\cos t - \frac{3\pi}{4} - \frac{3\pi}{4}\sin t} + \sin\left(\frac{3\pi}{4}\cos t + \frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\sin t\right)$$

Knowing that $h(0) = f(B) > 0$ and $h(\pi) = f(A) < 0$, we can find a root t of h on the interval $[0, \pi]$. This root corresponds to a root $(x(t), y(t))$ of f on the semicircle.

* Strength of bisection method:

- Simple to implement: requires little information about f.
- Easy to estimate error.

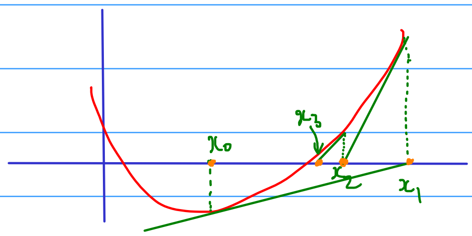
* Weaknesses:

- Convergence is generally slower than other methods.
- Doesn't take advantage of helpful information about f.

Newton's method (1660s)

Find root of $f(x)=0$.

While this method can work for multivariable x, its idea is most intuitive when x is a single variable.



- Take a point x_0 as the initial iteration.
- Draw a tangent line to the graph of f at $(x_0, f(x_0))$.
- Find the intersection between this tangent line and the x-axis. Call the intersection x_1.
- Repeat this procedure.

* Let's write this idea rigorous:

The equation of the tangent line at x_0 is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The intersection with the x-axis is obtained by setting $y=0$:

$$-f(x_0) = f'(x_0)(x - x_0)$$

$$\rightsquigarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Therefore,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (*)$$

* Algorithm:

1) Start with some x_0.
2) Compute consecutively $x_1, x_2, x_3, \dots$ by the formula $(*)$.

Ex: Approximate $\sqrt{2}$ by Newton's method.

That is to find root of $f(x) = x^2 - 2$.

$$f'(x) = 2x$$

Then the iterative equation $(*)$ can be written as

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

Let's start with $x_0 = 4$.

Then $x_1 = \frac{4}{2} + \frac{1}{2} = 2.5$

$$x_2 = \frac{2.5}{2} + \frac{1}{2.5} = \cdots$$

$$x_3 = \cdots$$