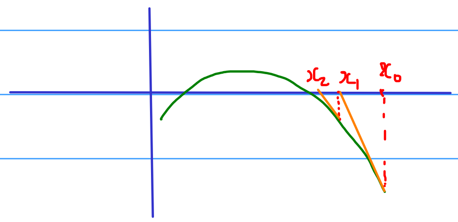
Lecture 11 (10/18/2019)

Newton method:

To solve for x from $f(x)=0$, we set up an iteration

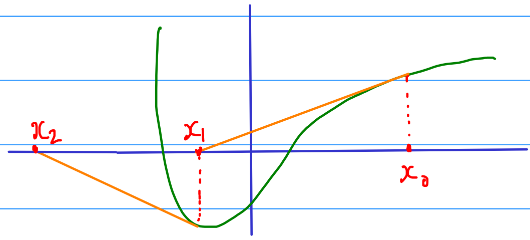
- Pick x_0 (close to exact solution)
- $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

f' serves as a guide to bring x_n's close to a solution.

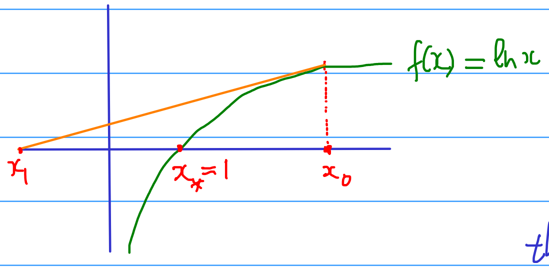


In general, Newton's method is fast. It fails when $f'(x_n)$ is equal to 0. It is not efficient when $f'(x_n)$ is too close to 0.
Some disadvantages of Newton's method include:

1) The first few iteration may be "wild", i.e. there may be big changes from x_0 to x_1, from x_1 to x_2, causing to sequence to approach another root (not a desired root).



2) The iteration might not be well-defined when $x_n - \frac{f(x_n)}{f'(x_n)}$ falls



outside of the domain of f.
Thus, unlike bisection method, Newton's method doesn't guarantee that we will get the root we want.

3) It's difficult to find an "a priori" error. This is mainly because Newton's method doesn't guarantee the sequence x_n has a limit.

An a priori error is the type of error that can be computed based on the given data at the beginning.
In bisection method, the given data are a, b and sometimes n (number of iterations) or ε (prescribe error). Recall

$$|x_* - c_n| \le \underbrace{\frac{b_0 - a_0}{2^{n+1}}}_{\text{known even before doing the iteration.}} \qquad (*)$$

Bisection method guarantees success:

- We know that we will find an approximate root in the given range $[a,b]$.
- We can control the error by deciding how many iterations to do.

For this reason, it is not surprised that one can find an a priori estimate such as $(*)$. Newton's method doesn't guarantee success:

- the sequence x_n may not converge.
- If x_n does converge, it may not converge to the root we expected.

For this reason, it's generally not possible to find such an a priori estimate as $(*)$.

* How to compute $\sqrt[3]{3}$ approximately $\underbrace{\text{by hand}}_{\text{using only } +,-,\times,/}$?

We have learned three ways to do it.

1) By Taylor polynomials:

$$f(x) = \sqrt[3]{x}$$

We know $f(1)$ and also all derivatives of f at 1.

$$f(3) = \underbrace{P_n(3)}_{\substack{\text{Taylor poly,} \\ \text{can be computed} \\ \text{by hand}}} + \underbrace{R_n(3)}_{\substack{\text{error term, can be} \\ \text{estimated by Lagrange theorem.}}}$$

$$\sqrt[3]{3} \approx P_n(3)$$

2) By Newton's method:

$$f(x) = x^3 - 3$$

Take x_0 close to $\sqrt[3]{3}$. We know that

$$\underbrace{\sqrt[3]{1}}_{1} < \sqrt[3]{3} < \underbrace{\sqrt[3]{8}}_{2}$$

Good candidates for x_0 include $1, 2, 1.5$.

Then do the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \underbrace{x_n - \frac{x_n^3 - 3}{3x_n^2}}_{A} = \underbrace{\frac{2x_n}{3} + \frac{1}{x_n^2}}_{B}$$

Note that B requires less operations to do than A.
B can be computed by hand.

3) Bisection method:

Choose $[a_0, b_0] = [1, 2]$. Then follow the bisection iteration.

More practice is on worksheet.