Lecture 12 (10/21/2019)

Last time, we discussed that it is difficult to estimate the error coming from Newton's method. This is caused by the fact that Newton's method doesn't guarantee success. In other words, the sequence defined by

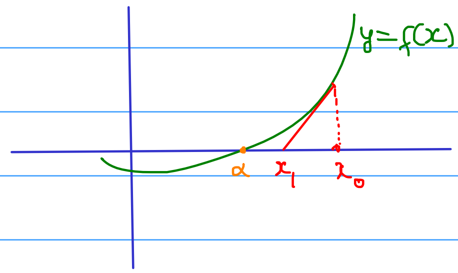
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

may not converge. (See Homework 4, problem 7).
In order to estimate the error $|x_n - \alpha|$ (where α is a true root), one must assume that x_n converges to α.

Note that one doesn't need this assume in bisection method. As the bisection procedure continues, the width $b_n - a_n$ of the interval $[a_n, b_n]$ keeps reducing by half. The limits of a_n and b_n must exist and equal to each other. The sequence of intervals $[a_n, b_n]$ is called a nesting sequence. The limit of $x_n = c_n$ exists as a consequence.

Back to Newton's method, we want to estimate $|x_n - \alpha|$. We have

$$x_{n+1} - \alpha = x_n - \alpha - \frac{f(x_n)}{f'(x_n)}. \qquad (*)$$



The problem is to estimate the quotient $\dfrac{f(x_n)}{f'(x_n)}$ as n is large.

Notice that α is the intersection of the graph of f and the x-axis. The point x_1 is the intersection of a tangent line of the graph (which is an approximation of the graph near α)

and the x-axis. This observation hints that first order Taylor approximation (or linear approximation) is used. Let us look at the first order Taylor expansion of f near α.

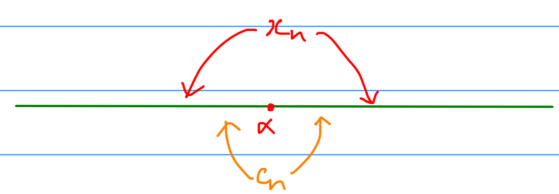
$$f(x) = \underbrace{f(\alpha) + f'(\alpha)(x-\alpha)}_{p_1(x)} + R_1(x)$$

By Lagrange theorem, there exists c between α and x such that
$$R_1(x) = \frac{f''(c)}{2!}(x-\alpha)^2 = \frac{f''(c)}{2}(x-\alpha)^2.$$

Thus,
$$f(x) = \underbrace{f(\alpha) + f'(\alpha)(x-\alpha)}_{p_1(x)} + \frac{f''(c)}{2}(x-\alpha)^2.$$

Replace x by x_n. Note that c depends on x_n (thus depending on n).

$$f(x_n) = \underbrace{f(\alpha)}_{=0} + f'(\alpha)(x_n-\alpha) + \frac{f''(c_n)}{2}(x_n-\alpha)^2$$



c_n lies between x_n and α.

Divide both sides by $f'(x_n)$:
$$\frac{f(x_n)}{f'(x_n)} = (x_n-\alpha)\frac{f'(\alpha)}{f'(x_n)} + \frac{f''(c_n)}{2f'(x_n)}(x_n-\alpha)^2$$

Because of the assumption that x_n converges to α,
$$\frac{f'(\alpha)}{f'(x_n)} \approx 1, \qquad \frac{f''(c_n)}{2f'(x_n)} \approx \frac{f''(\alpha)}{2f'(\alpha)}$$

Let's substitute these estimates into equation $(*)$:

$$x_{n+1} - \alpha = x_n - \alpha - \frac{f(x_n)}{f'(x_n)}$$

$$\simeq x_n - \alpha - (x_n - \alpha) - \frac{f''(\alpha)}{2f'(\alpha)}(x_n - \alpha)^2$$

$$= \underbrace{-\frac{f''(\alpha)}{2f'(\alpha)}}_{M}(x_n - a)^2$$

Thus, $\quad x_{n+1} - \alpha \simeq M(x_n - \alpha)^2$

Definition:

Consider a sequence x_n converging to some number α. If $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p$ for all large n, then p is called an order of the convergence. If $p = 1$ then C is called a linear rate of convergence.

Newton's method has order of convergence 2.

Bisection method satisfies

$$|\underbrace{x_n}_{c_n} - \alpha| \leq \frac{b_n - a_n}{2}$$

$$|x_{n+1} - \alpha| \leq \frac{b_{n+1} - a_{n+1}}{2} = \frac{1}{2}\,\frac{b_n - a_n}{2}$$

Of course these two estimates does not imply $|x_{n+1} - \alpha| \leq \frac{1}{2}|x_n - \alpha|$. However, the bisection method in general has order of convergence 1 and linear rate of convergence $1/2$. This observation should be taken as a consequence of heuristic argument, not as a fact (since it is not always true). The rate of convergence also depend on the given function f.