Lecture 13 (10/23/2019)

* Recall: if a sequence x_n converges to α and
$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p \text{ for all large } n$$
then p is called an order of convergence. If $p = 1$ then C is called a linear rate of convergence.

* Note that if p is an order of convergence then any $0 < q < p$ is also an order of convergence. Why?
$$|x_{n+1} - \alpha| \leq \underbrace{C|x_n - \alpha|^p \leq C|x_n - \alpha|^q}_{\substack{\text{true because} \\ |x_n - \alpha| \leq 1 \text{ for large } n}}$$
Thus, if x_n converges to α with order of convergence 2, it also converges to α with order of convergence 1.
The optimal order of convergence is the largest order of convergence (if such exists).

* If C is a linear rate of convergence then any $C' > C$ is also a linear rate of convergence. Indeed,
$$|x_{n+1} - \alpha| \leq C|x_n - \alpha| \leq C'|x_n - \alpha|$$
The optimal linear rate of convergence is the largest linear rate of convergence (if such exists).

* The name "linear rate of convergence" is used in this textbook, but it is not a standard term. Often, people use the term "rate of convergence" quite loosely to mean different things.
For example, if
$$|x_n - \alpha| \leq \phi(n) \quad \text{where } \phi(n) \to 0 \text{ as } n \to \infty$$
then people may call $\phi(n)$ a rate of convergence of x_n to α.

Examples of $\phi(n)$ are $2^{-n}, \frac{1}{n}, \frac{1}{n^2}, \frac{1}{n!}, \frac{1}{\log n}, \ldots$

* A converge of order $p>1$ is much faster than a convergence of order $p=1$. We can explain this fact as follows.

- If x_n converges to α with order p, then

$$|x_{n+1}-\alpha| \leq C\,|x_n-\alpha|^p \qquad (1)$$

for all n large. Because we are only interested the effect of p on the speed of convergence, let's assume that $C=1$ and that (1) holds for all n (not just for large n).

We apply (1) repeatedly:

$$\begin{aligned} |x_{n+1}-\alpha| \leq |x_n-\alpha|^p \leq \left(|x_{n-1}-\alpha|^p\right)^p &= |x_{n-1}-\alpha|^{p^2} \\ &\leq \left(|x_{n-2}-\alpha|^p\right)^{p^2} \\ &= |x_{n-2}-\alpha|^{p^3} \\ &\leq \ldots \\ &\leq |x_0-\alpha|^{p^{n+1}} \end{aligned}$$

This relation can be rewritten as

$$|x_n-\alpha| \leq |x_0-\alpha|^{p^n}$$

This is a relation between the error at the n'th step and the original error $|x_0-\alpha|$. If x_0 was chosen relatively close to α such that $|x_0-\alpha|<1$, then $|x_0-\alpha|^{p^n}$ goes to zero very rapidly. Note that the exponent p^n goes to ∞ quickly, making $|x_0-\alpha|^{p^n}$ decay to 0 evenly more quickly.

- If x_n converges to α with order $p=1$, linear rate of convergence C. Then

$$\begin{aligned} |x_{n+1}-\alpha| \leq C|x_n-\alpha| \leq C^2|x_{n-1}-\alpha| \leq C^3\,|x_{n-2}-\alpha| \\ \leq \ldots \\ \leq C^{n+1}\,|x_0-\alpha|. \end{aligned}$$

This relation can be rewritten as

$$|x_n - \alpha| \leq C^n |x_0 - \alpha|$$

The error at nth step is made small because of C (the linear rate of convergence). If $C < 1$ then C^n decays quickly to 0. The error $|x_n - \alpha|$ goes to 0 like C^n.

We see that this convergence is not as fast as the previous case. Think of $(0.5)^n$ versus $(0.3)^{2^n}$.

See examples on computing order of convergence on the worksheet.